Exhibit 11.3

Delta Hedge—Example 2

DELTA HEDGING **14-Aug-96**

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($19,554)
NET	$2,541

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%

OPTION PREMIUM	$2.1928	$2.7542	$1.1014	$1.3729	$2.5000
DELTA	0.5071	0.6273	0.4048	0.5758	1.0000
GAMMA	0.0714	0.0770	0.0996	0.1404	0.0000
THETA (per day)	0.0385	0.0427	0.0532	0.0773	0.0000
VEGA	0.1096	0.0913	0.0749	0.0544	0.0000
RHO	−0.0017	−0.0016	−0.0004	−0.0003	0.0000

Exhibit 11.3—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	6,273	4,048	5,758	10,000
HEDGE TRANSACTIONS	AMOUNT	PRICE					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$405,935	$405,935	$405,935
PURCHASE ASSETS	1,202	$101.50		$121,980			
PURCHASE ASSETS	(2,225)	$99.00					
PURCHASE ASSETS	1,710	$100.50				$171,860	$171,860
PURCHASE ASSETS	4,242	$102.50					$434,829
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	6,273	4,048	5,758	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.29	$100.29	$100.35	$101.26
TOTAL			$507,143	$629,123	$405,935	$577,795	$1,012,624
GAIN ON HEDGE ADJUSTMENT					($2,864.99)		($12,624)
INTEREST COST				($972.60)	($1,206.54)	($778.51)	($1,108.10)
CUMULATIVE COST				($972.60)	($5,044.13)	($5,822.63)	($19,554.48)

Exhibit 11.3—*continued*

DELTA HEDGING **14-Aug-96**

CALL OPTION EXAMPLE (OPTION EXPIRES OUT-OF-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($16,121)
NET	$5,975

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.00	$99.00	$100.50	$98.50
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%

OPTION PREMIUM	$2.1928	$2.4503	$1.1014	$1.3729	$0.0000
DELTA	0.5071	0.5881	0.4048	0.5758	0.0000
GAMMA	0.0714	0.0797	0.0996	0.1404	0.0000
THETA (per day)	0.0385	0.0439	0.0532	0.0773	0.0000
VEGA	0.1096	0.0936	0.0749	0.0544	0.0000
RHO	−0.0017	−0.0014	−0.0004	−0.0003	0.0000

Exhibit 11.3—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	5,881	4,048	5,758	0
HEDGE TRANSACTIONS	AMOUNT	PRICE					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$405,330	$405,330	
PURCHASE ASSETS	810	$101.00		$81,773			
PURCHASE ASSETS	(1,833)	$99.00					
PURCHASE ASSETS	1,710	$100.50				$171,860	
PURCHASE ASSETS	(5,758)	$98.50					
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	5,881	4,048	5,758	0
AVERAGE VALUE OF ASSETS			$100.00	$100.14	$100.14	$100.25	
TOTAL			$507,143	$588,916	$405,330	$577,189	$0
GAIN ON HEDGE ADJUSTMENT					($2,085.73)		($10,048.92)
INTEREST COST				($972.60)	($1,129.43)	($777.34)	($1,106.94)
CUMULATIVE COST				($972.60)	($4,187.76)	($4,965.10)	($16,120.96)

Exhibit 11.3—*continued*

DELTA HEDGING **14-Aug-96**

PUT OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($21,614)
NET	$482

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$98.00	$97.00	$100.50	$97.00
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%

OPTION PREMIUM	$2.1928	$3.0425	$3.4689	$0.8739	$3.0000
DELTA	−0.4852	−0.6506	−0.7728	−0.4223	−1.0000
GAMMA	0.0714	0.0780	0.0785	0.1404	0.0000
THETA (per day)	0.0385	0.0402	0.0395	0.0775	0.0000
VEGA	0.1096	0.0862	0.0566	0.0544	0.0000
RHO	−0.0017	−0.0018	−0.0013	−0.0002	0.0000

Exhibit 11.3—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			(4,852)	(6,506)	(7,728)	(4,223)	(10,000)
HEDGE TRANSACTIONS	**AMOUNT**	**PRICE**					
PURCHASE ASSETS	(4,852)	$100.00	($485,215)	($485,215)	($418,496)	($418,496)	
PURCHASE ASSETS	(1,654)	$98.00		($162,092)	($162,092)		
PURCHASE ASSETS	(1,222)	$97.00			($118,536)		
PURCHASE ASSETS	3,505	$100.50					
PURCHASE ASSETS	(5,777)	$97.00					($560,362)
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			(4,852)	(6,506)	(7,728)	(4,223)	(10,000)
AVERAGE VALUE OF ASSETS			$100.00	$99.49	$99.10	$99.10	$97.89
TOTAL			($485,215)	($647,307)	($765,843)	($418,496)	($978,858)
GAIN ON HEDGE ADJUSTMENT						($4,915.63)	($21,141.77)
INTEREST COST				$930.55	$1,241.41	$1,468.74	$802.60
CUMULATIVE COST				$930.55	$2,171.96	($1,274.92)	($21,614.10)

Exhibit 11.3—*continued*

DELTA HEDGING **14-Aug-96**

PUT OPTION EXAMPLE (OPTION EXPIRES OUT-OF-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($12,729)
NET	$9,366

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$98.00	$97.00	$100.50	$102.00
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%

OPTION PREMIUM	$2.1928	$3.0425	$3.4689	$0.8739	$0.0000
DELTA	−0.4852	−0.6506	−0.7728	−0.4223	0.0000
GAMMA	0.0714	0.0780	0.0785	0.1404	0.0000
THETA (per day)	0.0385	0.0402	0.0395	0.0775	0.0000
VEGA	0.1096	0.0862	0.0566	0.0544	0.0000
RHO	−0.0017	−0.0018	−0.0013	−0.0002	0.0000

Exhibit 11.3—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			(4,852)	(6,506)	(7,728)	(4,223)	0
HEDGE TRANSACTIONS	**AMOUNT**	**PRICE**					
PURCHASE ASSETS	(4,852)	$100.00	($485,215)	($485,215)	($485,215)	($418,496)	
PURCHASE ASSETS	(1,654)	$98.00		($162,092)	($162,092)		
PURCHASE ASSETS	(1,222)	$97.00			($118,536)		
PURCHASE ASSETS	3,505	$100.50					
PURCHASE ASSETS	4,223	$102.00					
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			(4,852)	(6,506)	(7,728)	(4,223)	0
AVERAGE VALUE OF ASSETS			$100.00	$99.49	$99.10	$99.10	
TOTAL			($485,215)	($647,307)	($765,843)	($418,496)	$0
GAIN ON HEDGE ADJUSTMENT						($4,915.63)	($12,257.12)
INTEREST COST				$930.55	$1,241.41	($1,468.74	$802.60
CUMULATIVE COST				$930.55	$2,171.96	($1,274.92)	($12,729.45)

Four examples are included: replication of a call option which expires in-the-money; replication of a call option which expires out-of-the-money; replication of a put option which expires in-the-money; and replication of a put option which expires out-of-the-money. The examples entail the assumption that the hedges are rebalanced at the end of every seven days (an arbitrary choice). In the following discussion, the focus is on the replication of the call option which expires in-the-money. However, the issues in respect of the other cases are similar.

As is evident, the hedge profit and loss on this option show a net gain of $2,541. The gain is calculated as the premium receipt, including interest on premium, adjusted for the hedging costs, which incorporates trading gains and losses and the cost of financing the asset positions.

If the option replication position had been consistent with the theory, then the net hedge profit and loss should have been zero; that is, the cost of synthesising the option should have been exactly equal to the option premium. In reality, this will rarely be the case. The difference reflects the following factors:

- actual experienced volatility of asset price changes relative to the volatility implicit in the option premium; and
- inefficiencies in the hedge portfolio and the hedge process.

In this case, the actual experienced volatility of asset price changes was approximately 18.02% pa. This compares to the volatility used to price the option of 20.00% pa. *Exhibit 11.4* sets out the performance of the hedge portfolio where the actual experienced volatility is used to price the option. As would be expected, the net hedge profit and loss is zero. Note also the differences in the hedge requirements, reflecting the changes in the option delta arising from the lower volatility (see more detailed discussion below).

The second factor which may affect the hedge profit and loss is the inefficiencies in the hedge portfolio and the hedging process. This is caused by a number of factors, including the frequency of rebalancing and the presence of discontinuities or gaps in the asset price path, or non-constancy of volatility and/or interest rates.

As noted, the frequency of rebalancing will influence the efficiency of the hedge. As the objective of the hedge is to maintain delta neutrality, each time the asset price changes, the delta of the option will also change, requiring rebalancing of the hedge. However, the presence of transaction costs dictates that the hedge only be readjusted *periodically*. This is designed to minimise the transaction costs while maintaining delta matching to a sufficient degree. This process of periodic rather than continuous rebalancing necessarily means that the trader will have small exposures to the *asset price changes* creating gains and losses which vary from the theoretical option premium.

A further source of hedge costs results from the changes in delta itself. As notes, changes in delta require adjustments in the hedge. However, if the change in the asset price is large, in particular, a discontinuity or sharp jump, the lag in adjusting the hedge will necessarily mean a divergence between the replication costs and the theoretical value of the option. In effect, the delta hedge, as discussed in detail below, does not cover the gamma risk of the strategy.

The theoretical assumptions that both volatility and interest rates remain constant are unlikely to be realised in practice. The non-constancy of these parameters will also affect the performance of the hedge.

Changes in volatility will affect the hedging process in two separate ways:

1. The first impact will be that the movement in volatility will impact upon the value of the option but not the value of the asset position.
2. The second impact will be that the change in volatility will alter the option delta and will thereby affect the costs of the hedge.

The impact of volatility changes is illustrated by the examples in *Exhibit 11.5*. In this example, the hedge performance of the call option is considered where the volatility increases from the original level (20%) to 25% or decreases to 15%. In each case the change occurs after one week and the volatility remains at the higher or lower level for the remainder of the life of the option.

Exhibit 11.4

Delta Hedge—Based on Actual Volatility

DELTA HEDGING

26-Aug-96

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	18.02%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$19,758
INTEREST ON PREMIUM	$152
HEDGE COSTS	($19,910)
NET	$0

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	18.02%	18.02%	18.02%	18.02%	18.02%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%

OPTION PREMIUM	$1.9758	$2.5744	$0.9536	$1.2655	$2.5000
DELTA	0.5061	0.6392	0.3932	0.5830	1.0000
GAMMA	0.0793	0.0845	0.1098	0.1552	0.0000
THETA (per day)	0.0347	0.0380	0.0476	0.0694	0.0000
VEGA	0.1096	0.0903	0.0744	0.0542	0.0000
RHO	−0.0015	−0.0015	−0.0004	−0.0002	0.0000

Exhibit 11.4—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,061	6,392	3,932	5,830	10,000
HEDGE TRANSACTIONS	AMOUNT	PRICE					
PURCHASE ASSETS	5,061	$100.00	$506,058	$506,058	$394,405	$394,405	$394,405
PURCHASE ASSETS	1,331	$101.50		$135,089			
PURCHASE ASSETS	(2,460)	$99.00					
PURCHASE ASSETS	1,898	$100.50				$190,728	$190,728
PURCHASE ASSETS	4,170	$102.50					$427,469
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,061	6,392	3,932	5,830	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.31	$100.31	$100.37	$101.26
TOTAL			$506,058	$641,147	$394,405	$585,134	$1,012,603
GAIN ON HEDGE ADJUSTMENT					($3,228.03)		($12,603)
INTEREST COST				($970.52)	($1,229.60)	($756.39)	($1,122.17)
CUMULATIVE COST				($970.52)	($5,428.15)	($6,184.54)	($19,909.78)

As is evident, the value of the option changes as a result of the changes in volatility levels. The impacts of these changes are only unrealised gains or losses where the option is marked-to-market. There is no realised or cash impact of the change in volatility *unless the option is sold or repurchased.* Where the option position is hedged with a dynamically managed position in the asset, changes in volatility have no direct *cash* impact on the cost of replicating the option other than through the impact on the option deltas.

The change in volatility impacts on the option deltas. An increase (decrease) in volatility decreases (increases) the option delta. This is apparent in *Exhibit 11.5*. The altered position in the asset affects the hedging costs.

Exhibit 11.5

Delta Hedge—Performance Where Volatility is not Constant

DELTA HEDGING 26-Aug-96

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($18,927)
NET	$3,169

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	20.00%	25.00%	25.00%	25.00%	25.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%
OPTION PREMIUM	$2.1928	$3.2152	$1.4784	$1.6457	$2.5000
DELTA	0.5071	0.6061	0.4266	0.5630	1.0000
GAMMA	0.0714	0.0627	0.0807	0.1130	0.0000
THETA (per day)	0.0385	0.0544	0.0673	0.0972	0.0000
VEGA	0.1096	0.0929	0.0758	0.0547	0.0000
RHO	−0.0017	−0.0018	−0.0006	−0.0003	0.0000

Exhibit 11.5—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	6,061	4,266	5,630	10,000
HEDGE TRANSACTIONS	**AMOUNT**	**PRICE**					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$427,672	$427,672	$427,672
PURCHASE ASSETS	990	$101.50		$100,441			
PURCHASE ASSETS	(1,795)	$99.00					
PURCHASE ASSETS	1,364	$100.50				$137,073	$137,073
PURCHASE ASSETS	4,370	$102.50					$447,906
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	6,061	4,266	5,630	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.24	$100.24	$100.31	$101.27
TOTAL			$507,143	$607,584	$427,672	$564,745	$1,012,651
GAIN ON HEDGE ADJUSTMENT					($2,234.25)		($12,651)
INTEREST COST				($972.60)	($1,165.23)	($820.19)	($1,083.07)
CUMULATIVE COST				($972.60)	($4,372.09)	($5,192.28)	($18,926.66)

Exhibit 11.5—*continued*

DELTA HEDGING **26-Aug-96**

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($20,578)
NET	$1,518

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	20.00%	15.00%	15.00%	15.00%	15.00%
INTEREST RATE (CURRENT)	10.00%	10.00%	10.00%	10.00%	10.00%
OPTION PREMIUM	$2.1928	$2.3048	$0.7308	$1.1025	$2.5000
DELTA	0.5071	0.6632	0.3702	0.5977	1.0000
GAMMA	0.0714	0.0989	0.1295	0.1848	0.0000
THETA (per day)	0.0385	0.0308	0.0389	0.0572	0.0000
VEGA	0.1096	0.0880	0.0730	0.0537	0.0000
RHO	−0.0017	−0.0013	−0.0003	−0.0002	0.0000

Exhibit 11.5—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	6,632	3,702	5,977	10,000
HEDGE TRANSACTIONS	**AMOUNT**	**PRICE**					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$371,557	$371,557	$371,557
PURCHASE ASSETS	1,561	$101.50		$158,428			
PURCHASE ASSETS	(2,930)	$99.00					
PURCHASE ASSETS	2,275	$100.50				$228,635	$228,635
PURCHASE ASSETS	4,023	$102.50					$412,309
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	6,632	3,702	5,977	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.35	$100.35	$100.41	$101.25
TOTAL			$507,143	$665,571	$371,557	$600,192	$1,012,501
GAIN ON HEDGE ADJUSTMENT					($3,964.05)		($12,501)
INTEREST COST				($972.60)	($1,276.44)	($712.57)	($1,151.05)
CUMULATIVE COST				($972.60)	($6,213.09)	($6,925.67)	($20,577.57)

The change in interest rates impacts on the cost of replication in two ways:

1. The first is its impact on the option delta, although this is relatively insignificant.
2. The second and more significant impact is by way of higher funding costs in financing the asset holding, or higher investment returns on the proceeds of the short sale.

Exhibit 11.6 sets out an example of the impact of interest rate changes on the process of replication for both increases or decreases.

Exhibit 11.6

Delta Hedge—Performance Where Interest Rates are not Constant

DELTA HEDGING 26-Aug-96

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($20,356)
NET	$1,740

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	12.00%	12.00%	12.00%	12.00%

OPTION PREMIUM	$2.1928	$2.7510	$1.1005	$1.3724	$2.5000
DELTA	0.5071	0.6266	0.4045	0.5756	1.0000
GAMMA	0.0714	0.0769	0.0996	0.1403	0.0000
THETA (per day)	0.0385	0.0425	0.0531	0.0772	0.0000
VEGA	0.1096	0.0912	0.0749	0.0544	0.0000
RHO	−0.0017	−0.0016	−0.0004	−0.0003	0.0000

Exhibit 11.6—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	6,266	4,045	5,756	10,000
HEDGE TRANSACTIONS	AMOUNT	PRICE					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$405,619	$405,619	$405,619
PURCHASE ASSETS	1,195	$101.50		$121,247			
PURCHASE ASSETS	(2,221)	$99.00					
PURCHASE ASSETS	1,711	$100.50				$171,950	$171,950
PURCHASE ASSETS	4,244	$102.50					$435,055
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	6,266	4,045	5,756	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.29	$100.29	$100.35	$101.26
TOTAL			$507,143	$628,390	$405,619	$577,568	$1,012,623
GAIN ON HEDGE ADJUSTMENT					($2,856.59)		($12,623)
INTEREST COST				($1,167.12)	($1,466.16)	($933.48)	($1,329.20)
CUMULATIVE COST				($1,167.12)	($5,469.88)	($6,403.35)	($20,355.73)

Exhibit 11.6—*continued*

DELTA HEDGING **26-Aug-96**

CALL OPTION EXAMPLE (OPTION EXPIRES IN-THE-MONEY)

INPUTS	
ASSET PRICE	$100.00
STRIKE PRICE	$100.00
TRADE DATE	01-Jan-96
EXPIRY DATE	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00
VOLATILITY (INITIAL)	20.00%
INTEREST RATE	10.00%
HOLDING COST	0.00%

HEDGE PROFIT AND LOSS	
PREMIUM RECEIPT	$21,928
INTEREST ON PREMIUM	$168
HEDGE COSTS	($18,752)
NET	$3,344

NO OF ASSETS:	10,000	10,000	10,000	10,000	10,000
DATE	01-Jan-96	08-Jan-96	15-Jan-96	22-Jan-96	29-Jan-96
TIME TO EXPIRY (DAYS)	28.00	21.00	14.00	7.00	0.00
ASSET PRICE	$100.00	$101.50	$99.00	$100.50	$102.50
VOLATILITY (CURRENT)	20.00%	20.00%	20.00%	20.00%	20.00%
INTEREST RATE (CURRENT)	10.00%	8.00%	8.00%	8.00%	8.00%

OPTION PREMIUM	$2.1928	$2.7574	$1.1022	$1.3735	$2.5000
DELTA	0.5071	0.6280	0.4051	0.5760	1.0000
GAMMA	0.0714	0.0771	0.0997	0.1404	0.0000
THETA (per day)	0.0385	0.0429	0.0533	0.0774	0.0000
VEGA	0.1096	0.0914	0.0750	0.0544	0.0000
RHO	−0.0017	−0.0016	−0.0004	−0.0003	0.0000

Exhibit 11.6—*continued*

DELTA HEDGE							
HEDGE REQUIREMENT			5,071	6,280	4,051	5,760	10,000
HEDGE TRANSACTIONS	**AMOUNT**	**PRICE**					
PURCHASE ASSETS	5,071	$100.00	$507,143	$507,143	$406,253	$406,253	$406,253
PURCHASE ASSETS	1,209	$101.50		$122,713			
PURCHASE ASSETS	(2,230)	$99.00					
PURCHASE ASSETS	1,709	$100.50				$171,770	$171,770
PURCHASE ASSETS	4,240	$102.50					$434,602
VALUE OF HEDGE PORTFOLIO							
NO OF ASSETS			5,071	6,280	4,051	5,760	10,000
AVERAGE VALUE OF ASSETS			$100.00	$100.29	$100.29	$100.35	$101.26
TOTAL			$507,143	$629,856	$406,253	$578,022	$1,012,624
GAIN ON HEDGE ADJUSTMENT					($2,873.39)		($12,624)
INTEREST COST				($778.08)	($966.35)	($623.29)	($886.83)
CUMULATIVE COST				($778.08)	($4,617.83)	($5,241.12)	($18,752.27)

3. OPTION REPLICATION—RISK MANAGEMENT

3.1 Risk dimensions

The use of option replication techniques entails significant risks which require management. In practice, these risk are expressed, measured and managed using the sensitivities of options as depicted by the "Greeks" (see Chapter 10). The major risk dimensions include:

- delta/gamma, or exposure to asset price or changes in delta;
- vega/kappa, or volatility risk;
- theta, or time decay risk; and
- rho, or interest rate risk.

These Greek letters provide a measure of the behaviour of option value as market conditions change, and allow evaluation and management of the risk for an individual option or, more realistically, within an option portfolio.

In this section, the risk management process of option replication is considered through consideration of the management of each of the identified risk dimensions.

3.2 Delta-gamma risk

As previously outlined, the delta for derivative securities can be defined as the change of its price with respect to the price of the underlying asset. Delta, as a construct, emerges quite clearly from the Black-Scholes option pricing approach, which implies the possibility of establishing an instantaneous riskless portfolio consisting of a position in the derivative security and a position in the underlying asset.

The option's delta, as is evident from the discussion above, is essential to managing its risk within a hedge portfolio, in which the option is hedged by purchasing or selling delta units of the underlying asset. The risk management function (at least, as noted below, for small price changes) is determined by delta neutrality, whereby the overall portfolio of options and hedges has a delta of zero, thereby immunising the portfolio for changes in value from both price increases and decreases.

The concept of delta also enables one option to be hedged *with another option* on the same underlying asset by creating delta neutral positions. As both options are affected by movements in the price of the underlying asset, to ensure this is necessary to hold the options in the right proportions, in the resulting changes in value to offset.

The delta of the underlying asset is, by definition, 1.0. In practice, delta hedging utilises a futures or forward contract on the underlying asset rather than the asset itself. The principles of delta hedging are equally applicable where a futures or forward position in the underlying asset is utilised. The futures or forward contract utilised as the hedge does not, necessarily, have to have the same maturity as the derivative security. However, the use of futures or forward contracts introduces the fair pricing issues identified above.

Deltas have the property of additivity. The delta of a portfolio of options and other derivative securities on an asset is the sum of the deltas of the individual options and other assets in the portfolio. This facilitates the use of delta to summarise the price sensitivity of even a very complex portfolio of assets and derivative products.

However, delta only holds for very small changes in the asset price. Therefore, any movement in the asset price outside a small range leads to diminished hedge efficiency. This change in the asset price exposes the portfolio to what is commonly referred to as gap/jump risk or gamma risk.

Gamma is the change in delta as the underlying asset price changes. As changes in delta produce exposure in a portfolio, the level of gamma can be utilised to quantify the risk of an option portfolio. The gamma of an option is not stationary, but changes as the asset price changes. Gamma is highest when the option is at-the-money with a short time to expiry.

The basic problem of delta hedging, therefore, is that while the delta of an option varies substantially through its life, the delta of the underlying asset is always fixed. Consequently, in theory, continual rehedging is required to keep the portfolio perfectly delta neutral. This problem is sometimes sought to be averted by a technique known as gamma hedging, whereby the intermediary seeks to match the rate at which the deltas themselves vary with changes in asset prices.

The concept of gamma neutral hedging recognises that adjusting the portfolio to delta neutrality after each asset price change does not give adequate protection to a portfolio. A gamma hedge is a hedge strategy that attempts to reduce the exposure of the portfolio by reducing total portfolio gamma.

A portfolio which is fully hedged will have both a zero delta and a zero gamma. The only means of creating a zero gamma position is to match each option with the offsetting position in that option series. Consequently, gamma is sought to be minimised by changing the composition of the option component of the portfolio. Changing the underlying asset side of the portfolio has no impact on gamma. For example, the gamma impact of selling at-the-money puts can be minimised by buying out-of-the-money puts.

Against this background, the notion of delta and gamma neutrality can be stated more precisely. A delta neutral option or book of options hedged with offsetting positions in the underlying asset is one whose value is unaffected by (small) changes in the price of that underlying asset. A gamma neutral option portfolio is one that remains delta neutral as the price of the underlying asset changes.

In practice, gamma indicates the extent of portfolio rebalancing that will be needed in a delta neutral position. A large gamma position indicates that a portfolio will require substantial rehedging when the asset price alters. In essence, gamma is a measure of the risk exposure of a hedge position that will emerge when the price of the underlying asset changes, particularly where it changes rapidly or in a discontinuous manner or is not or is unable to be adjusted instantaneously. This is particularly the case where the price movement in the underlying asset is non-stochastic, commonly referred to as a price jump. The use of gamma to measure and manage the hedge risk is designed to, in fact, adjust the hedging process for the fact that the position in the underlying asset cannot be adjusted continuously as required by theoretical delta hedging.

As noted above, the gamma risk of a portfolio can only be reduced by purchasing options. An alternative measure of managing the gamma exposure on a portfolio is to restructure the portfolio configuration of options to replicate synthetic positions in the underlying asset itself. That is, create purchased/sold positions in the relevant series of call and put options to create purchased or sold position equivalents in the underlying asset, which by definition have limited gamma exposure.

3.3 Volatility risk

As previously noted, option values are particularly sensitive to changes in the volatility of the underlying asset. This is referred to as vega or volatility risk.

Volatility risk is a particularly important factor in option portfolio management. This reflects the fact that portfolio management, is, in part, an exercise in management of volatility positions. Portfolio managers converse in terms of being short or long volatility. The underlying premise is that the portfolio manager's task is to attempt to position the portfolio volatility exposure to seek to profit from changes in volatility level within pre-specified limits.

An important aspect of managing volatility risk is that it can only be neutralised by taking positions in the same or a different series of options. Volatility is not a determinant of value of the underlying asset and consequently cannot be neutralised through positions in the asset market itself.

3.4 Theta risk

The option theta measures the rate of change in the value of the portfolio with respect to time, where all other value parameters are held constant. In effect, theta measures the rate of time decay for the options or the degree to which it loses its inherent value as a "wasting asset". Theta measures the cost of holding an option and, conversely, the reward of selling an option.

As discussed below, management of theta is particularly important and its interaction with gamma and vega risks constitute a major component of option portfolio management.

3.5 Interest rate risk

Rho measures option portfolio risk with respect to changes in the riskfree interest rate. As previously outlined, the time value for a call option comes partly from the interest that can be earned investing the strike price from the present until the expiration date. Conversely, the purchaser of put loses interest while waiting until option maturity to receive the strike price. Consequently, changes in interest rates will have an impact on the value of the option.

In practice, the rho or riskless interest rate risk of the portfolio can be managed by assuming positions in the underlying interest rate market by traditional interest rate risk management measures.

3.6 Risk interactions

As might be expected, individual risk dimensions identified are substantively interrelated. This necessitates a constant process of trade-offs between the various risk dimensions and the management of an option portfolio. In this section, some key aspects of the trade-offs entailed in option portfolio management are examined.

For the trader, offsetting the price risk of the portfolio of options by delta-based purchases and sales of the underlying asset reduces the risk of the delta-neutral portfolio, whereby the value of the portfolio is insulated from the effects of small price changes in the underlying assets.

However, such a position still exposes the portfolio manager to significant risks. Delta-neutral positions entailing a net purchase of options will have a negative theta; that is, the option position will decline in value over time reflecting the nature of the option as a wasting asset. The value of the option evolves towards its intrinsic value as expiration approaches. In contrast, a delta-neutral position which entails sold options will gain in value as expiration approaches.

Interaction of delta and theta, in this context, dictates that market makers prefer delta-neutral positions that are net sold options so as to enable them earn the erosion in the value of the option; that is, benefit from the option portfolio theta. Structuring a portfolio to benefit from theta entails the portfolio manager assuming gamma and vega risk.

A portfolio which is delta-neutral and short options will require the portfolio manager to rebalance the hedge, in the case of a call by buying when the price of the underlying asset increases and selling the underlying asset where the price falls. Similarly, the portfolio is exposed to increases in value from falls in volatility, but loses value from increases in volatility.

To an extent, some of these risks are offsetting. For example, theta and gamma tend, at least to some degree, to offset the other. A net purchased option portfolio which is delta-neutral simultaneously loses value because of time decay or theta with the passage of time, but improves in value as a result of the movement in price because of the position's gamma risk. Conversely, a delta-neutral position entailing a net sold position in options increases in value because of theta each day, but, generally speaking, decreases in value when the market moves up and down because of the portfolio's gamma risk.

However, the risks are not necessarily entirely offsetting. Theta erosion in option values occurs relatively smoothly over the period to expiration. In contrast, the gains and losses from gamma rebalancing can be large and discontinuous, reflecting the fact that larger gains and losses from gamma occur when there is a sharp change in the price of the underlying asset which prevents the portfolio manager from adjusting or rebalancing the delta hedge appropriately.

The interaction of these risks typically forces portfolio managers to seek to generate earnings/value from an options portfolio by one or other of the following strategies:

1. Maintain delta neutral/net short options portfolio positions where it is anticipated that the underlying price movements will not be discontinuous (that is, the portfolio has low gamma risk) and volatility of the underlying assets are not expected to change substantially (that is, low vega/kappa risk).
2. Operate the portfolio on the basis that portfolio value/earnings derived from the bid-offer spread on purchasing and selling options. By implication, the portfolio manager would, under this strategy, seek to balance the portfolio by purchasing some options and selling others, in contrast to being a net purchaser or seller of options.
3. Managing the option portfolio as a volatility risk management function where portfolio managers seek to benefit from changes in volatility levels in the underlying asset.

The first strategy is fairly self-explanatory, and is consistent with the factors described above.

The second approach is also relatively straightforward and is predicated on the notion that portfolio managers view delta hedging in the underlying asset or option replication techniques as a *temporary* substitute for an offsetting option transaction. In effect, the portfolio manager plans to earn the bid-offer spread without seeking to worry about the erosion in value of the option portfolio or the management of the complex gamma and vega risks entailed.

The last approach, essentially taking views on volatility, seeks to generate profits according to the portfolio manager's capacity to anticipate price volatility changes for the underlying asset market, without taking a large market risk. The major problem for any option portfolio manager is that the volatility of the underlying asset, at least as it is utilised to calculate option values, has a significant level of uncertainty associated with its estimation. Consequently, the expected or true volatility of the asset can change over the life of the option, resulting in changes in portfolio valuation.

In practice, this dictates that portfolio managers may seek to balance the portfolio to earn the bid-offer spread, in volatility terms, whilst managing their portfolio to be delta, gamma, theta and kappa neutral. This is particularly the case where volatility levels in the underlying asset are subject to uncertainty or are experiencing rapid and sudden shifts. In contrast, in a market where volatility is "trending"' in one or other direction or the asset price is relatively stable, the portfolio manager may choose to take portfolio positions designed to facilitate increases in portfolio value in the event of the anticipated change in portfolio volatility occurring or from the benefit of accruing time value from selling options. Such positions regarding changes in volatility can be both for outright movements in volatility for a particular class of assets or positions for changes in the implied pattern of volatilities or its term structure.

3.7 Delta hedging in practice

The risk of using delta hedging techniques to replicate options can be assessed from the practice of portfolio insurance. This is a practice whereby asset managers create capital guaranteed (at a pre-agreed level) investments. This is done in two ways:

1. holding the asset and dynamically creating a put option on the asset to establish a minimum investment value; and
2. investing in cash and trading in asset to synthesise a call option on the asset to create exposure to increases in the asset price. The minimum portfolio value is represented by the total portfolio value adjusted for the cost of replicating the portfolio (effectively, the option premium).

This practice, which is quite widespread, performed poorly in the equity market crash in 1987.[5]

Similarly, option hedges, in currency, interest rate and commodity markets, performed below expectation in the ERM crises in 1992, the bond market collapse in 1994 and the Gulf War and the sharp fall in copper prices in 1996 following the disclosure of the Sumitomo copper trading losses.[6]

In each case, the failure of the dynamic hedge to efficiently replicate the option position sought to be hedged can be attributed to a combination of the following factors:

- the sharp, discontinuous jump in asset prices which made it difficult to maintain delta neutrality, creating exposure to the *asset price movements*;
- the sharp and unanticipated changes in volatility which also affected hedge performance; and
- the change in liquidity conditions in the underlying asset markets and the increase in transaction costs, which led to rises in the cost of the hedge over that which had been anticipated.

The use of futures contracts to replicate the option due to significant savings in transaction costs may have exacerbated the problems, as in these stressful conditions the pricing of the futures contracts may have deviated from fair value, creating further hedge slippage.

4. OPTION REPLICATION—DEVELOPMENTS

The risks of replicating options dynamically have led researchers to experiment with different strategies to improve the efficiency of the hedge. Three areas of development merit particular comment:

1. the hedging of short versus long dated options;
2. transaction cost incorporated strategies; and

5. See Jon Taylor and Matthew Smith, "Option Replication" (1987) (Dec) *Intermarket* 16-55; Desmond Fitzgerald and Janette Rutherford, "Variations On A Theme" (1988) 1(7) *Risk* 30-31.
6. For a perspective on practical issues in option hedging see Krystna Kryzak, "Gamma Raison" (1990) 3(5) *Risk* 21-27; Richard Cookson, "Models Of Imperfection" (1992) 5(9) *Risk* 55-61; Richard Cookson and Lilian Chew, "Things Fall Apart" (1992) (Oct) *Risk* 44-53.

3. static option replication techniques.

The distinction between hedging short versus long dated options is driven by the different risk characteristics. For short dated options, the typical option sensitivities (delta, gamma, vega, theta, and rho) may not provide accurate risk measures. This reflects the more significant impact of sharp movements in asset price and volatility in theses options.

This has led to the development of approaches to option portfolio management which segment the portfolio by remaining time to maturity. A typical segmentation would be short (under 30 days), medium (up to 6-12 months) and long (beyond 12 months).

Within this segmented framework, the principal, albeit not the sole, focus is as follows:

- for short options, gamma risk management is important;
- for medium to longer dated options, vega risk management is important.

This segmentation also has implications for the process of replication and the underlying instrument used. The risk of volatility changes for longer dated options favours the use of *options*, rather than the asset, to replicate these positions. This is discussed below in the context of static option replication.

As noted above, the presence of transaction costs impacts on the cost of the hedge, leading to a tradeoff between frequency of rehedging and the transaction costs incurred. The impact of transaction costs is that the trader must, of necessity, select a hedging strategy incorporating a discrete hedging algorithm. This has, as a by-product, implications for the valuation of the option.

A number of models have emerged which provide hedging strategies for the replication of options incorporating transaction costs. These models are based on placing boundaries on the risk of replication measured by the variance of the hedge portfolio. Alternative approaches use utility functions which relate both to the risk and return.[7]

Static portfolio replication entails the use of *options* rather than trading in the underlying asset to replicate the option. The major rationale for this type of replication is that it avoids the risk of underperformance of the hedge where the asset price moves are discontinuous and large or where there are shifts in volatility levels. The major application of static portfolio replication is in relation to longer or more complex options (including exotic options).

The basic approach is to model the option payoff (using an option pricing model) at maturity, using different asset prices and volatilities. The model payoff is then sought to be replicated by a portfolio of options (with different expiries and strike prices). The technique usually involves the use of some

7. For discussion on different strategies for optimising the cost of hedging options see Elizabeth Whalley and Paul Wilmott, "Counting The Cost" (1993) 6(10) *Risk* 59-66; Elizabeth Whalley and Paul Wilmott, "Hedge With An Edge" (1994) 7(10) *Risk* 82-85.

form of optimisation technique, such as multiple regression, to generate the portfolio of options which best replicates the option sought to be synthesised.[8]

5. SUMMARY

The potential to replicate options through trading in the underlying asset is inherent in the approach to valuation of options. By trading in the asset and either financing the investment or investing the proceeds of a short sale and adjusting the asset position as asset prices, volatilities, interest rates and time to maturity change, it is theoretically possible to replicate the economic profile of an option.

However, this process of delta hedging is only accurate for small movements in asset. In particular, sharp and discontinuous changes in asset prices, changes in volatility or changes in interest rates may expose the trader to the risk of the asset portfolio underperforming the option sought to be replicated. This dictates the use of sophisticated risk control mechanisms to monitor and manage the risk of the replicating portfolio to maintain the accuracy of the hedge and to match its performance to that of the option.

8. See Jon Frye, "Static Portfolio Replication" (1988) 1(11) *Risk* 22-23; Kenneth S Choie and Frederick Novomestky, "Replication Of Long Term With Short Term Options" (1989) (Winter) *Journal of Portfolio Management* 17-19; Emanuel Derman, Deniz Ergener and Iraj Kani, "Forever Hedged" (1994) 7(9) *Risk* 139-145.

Part 4

Investment Management

Chapter 12

Portfolio Optimisation

by Geoffrey Brianton

1. INTRODUCTION

This chapter will investigate portfolio optimisation. That is, the construction of portfolios that for some specific criteria are better than all other portfolios. The main area of study will be the model at the heart of Modern Portfolio Theory—the "mean-variance" model.

This model was first proposed in the 1950s by Harry Markowitz. Although his landmark paper is one of the most widely cited in finance, the mean-variance optimisation approach is not as widely used in practice as this would suggest. This chapter will focus on the reasons for this, and investigate some of the practical steps that can be taken to tame the model. It is written from the point of view of a fund manager.

The starting point will be a review of utility theory and risk aversion, as this is the basis for the "mean-variance" model.

2. UTILITY, RISK AVERSION AND PROBABILITY DISTRIBUTIONS

Imagine you are walking down the street and, to your delight, find a dollar on the pavement. Now, if instead of a single dollar you find two dollars, you would be even more delighted. If you found three dollars it would be an even better event. However, as the amount found increases, the additional joy of finding a larger amount of money becomes smaller. The satisfaction in finding $10,000 would be almost indistinguishable from that of finding $10,001. This is known as diminishing marginal utility. That is, at the margin, the utility (benefit) gained reduces with each additional dollar found.

One of the consequences of diminishing marginal utility is that investors are reticent to put at risk existing wealth in order to gain new wealth. This is because the last dollar is more valuable than the next one. Consider an investor who is faced with two options. One is a certain gain of $10, the other is a 50% chance of a $5 return and a 50% chance of gaining $15. Both options will return on average $10, so what would be the choice of an investor with diminishing margin utility? *Exhibit 12.1* shows the amount utility of a return of five, ten and fifteen dollars expressed in term of ticks.

Exhibit 12.1

Utility for Various Returns

Return amount	Utility
$5	✓
$10	✓✓✓
$15	✓✓✓✓

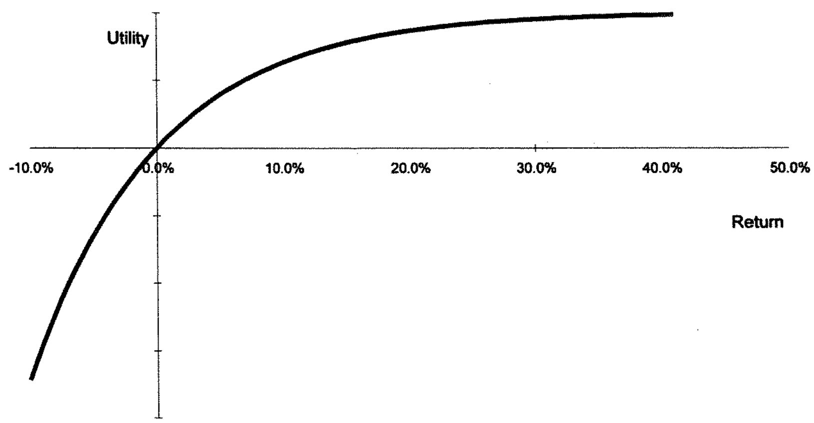
Under the first option ($10 with certainty) the investor has a gain of three ticks worth of utility. With the second option (50% change of $15 and 50% change of $5) the investor will receive either one or four ticks with an equal probability, an expected outcome of two and a half ticks. The investor would always choose the first option due to the extra half a tick of utility. The investor characteristics of choosing a certain return over an uncertain return when, on average, both will produce the same expected return is known as risk aversion. Risk aversion is a direct result of the next one being good, but not quite as good as the last one.

2.1 Utility functions

In solving the problem of portfolio construction, a more complete definition of an investor's utility than the number of ticks for three different returns is required. An investor's utility function defines the relationship between return and the level of utility received. *Exhibit 12.2* shows a typical utility function.

Exhibit 12.2

Utility as a Function of Return



On the horizontal axis is the return (in percentage terms), while the vertical axis indicates the investor's utility for that return. This utility function has three key characteristics that are common to all utility functions:

- it is always positive (known as non-satiation)
- it is upwardly sloping (more is better); and
- the slope decreases as the return increases (diminishing marginal utility).

In mathematical terms, the utility function has a positive first derivative and a negative second derivative. There are a wide variety of utility functions that fit the above criteria. The function used most frequently in portfolio choice problems is known as the exponential utility function. It has the form

$$U(r) = 1 - e^{-\lambda r}$$

Where

λ is a risk aversion parameter (pronounced lambda);

U is the level of utility for a given level of return; and

r is the return.

It has the advantages of having the appropriate properties of a utility function and, by varying the risk aversion parameter, it can model a full range of investor preferences from very risk averse to risk neutral.[1]

1. Ultimately there is no clear defining reason why the exponential utility function should be used in preference to other utility functions. As the exponential function is flexible and mathematically simple to manipulate, it is widely used but this does not mean that other functions are inappropriate.

Exhibit 12.3

A Range of Utility Functions

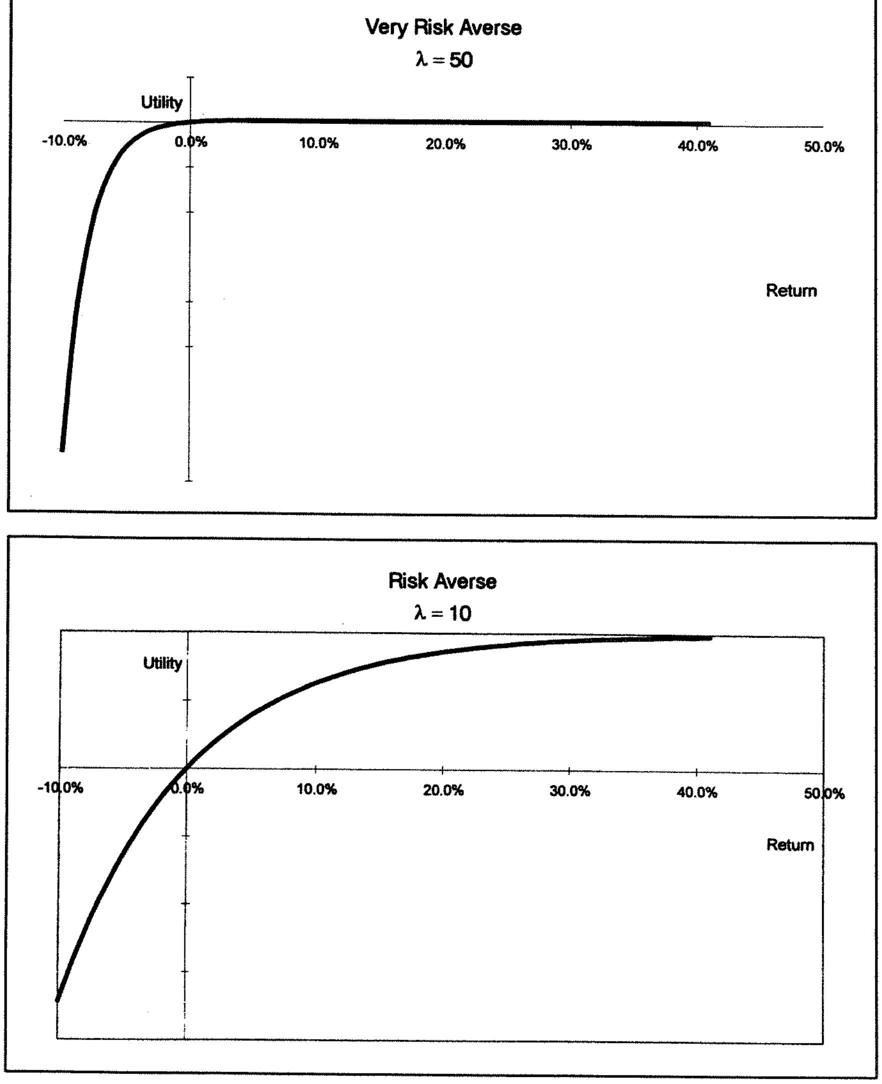
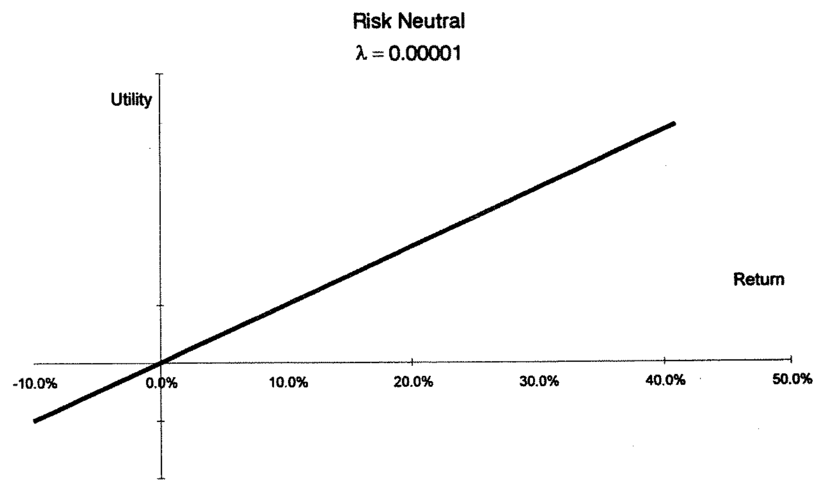


Exhibit 12.3—*continued*



The risk neutral investor treats each additional return as being equally desirable. As a result, this investor's choice is always the asset with highest expected return regardless of the potential losses. The "very risk averse" investor places a very low additional utility from higher returns and would strongly tend toward assets with stable returns and low probabilities of losses. The "risk averse" investor lies between these two.

2.2 What is your λ?—a two asset example

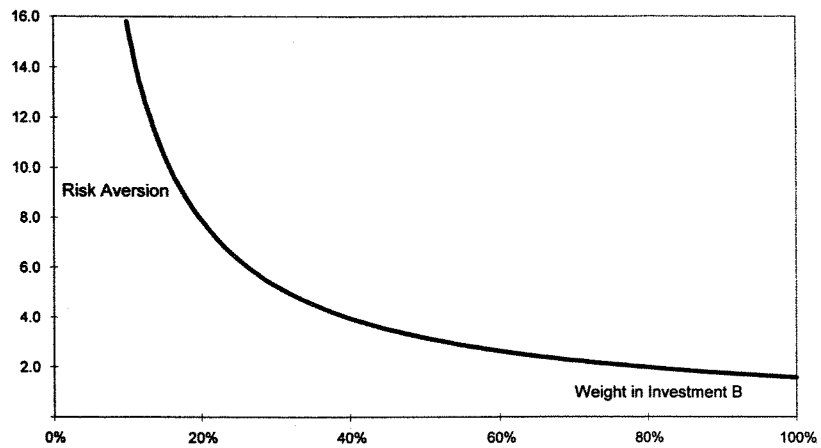
The appropriate risk aversion (λ) for a given investor is not an easy question. It is generally not possible to simply ask most investors "So what is your λ?" Typically an investor's risk aversion needs to be determined indirectly by examining their behaviour and choices. Consider the following example: each year an investor has the choice of two assets. Investment A has a certain return of 3%. Investment B will return either 40%, 10% or −20% with a probability of $^1/_4$, $^1/_2$ and $^1/_4$ respectively. On average, investment B will return 10%, significantly higher than A, but there is a one in four chance of a substantial negative return.

The amount an investor is prepared to put into the risky asset (B) indicates how venturesome is their investment approach. *Exhibit 12.4* shows the relationship between the weight in investment B and the λ that would make that choice optimal (if we assume the investor has an exponential utility function).[2] The larger the investment in the risky asset the less risk averse the investor. (The reader is encouraged to do this exercise from their own point of view.)

2. Appendix II gives the mathematical derivation of this relationship.

Exhibit 12.4

Relationship Between Weight in Risky Asset and Risk Aversion



For example, consider an investor saving for her retirement. She is in her early fifties and plans to work for another five years or so. She has a moderate level of wealth which (not surprisingly) she would like to grow. She is cautious about putting at risk the retirement lifestyle she has already secured. Assume that she has decided to put 30% of her portfolio in the risky assets. This would equate to a λ of about 5.

The task of modeling an investor's level of risk aversion is complex. However, for the purposes of this chapter we will continue assuming this simple two asset question and the example investor's response and see how far the problem of portfolio optimisation can be progressed.

2.3 Probability distributions

At this stage, half of the problem has been solved, that being a robust model of the investor's utility. A model for behaviour of the assets is also needed.

In the previous two-asset example, the return of asset B in the next period was not known but the likelihood of various returns was known. In statistical terms this is called the *probability distribution* of the returns from B. The most common approach used for asset returns is to model the log of the returns. Log returns $R = \ln(1 + r)$ (where r is the return and ln is the natural logarithm function) are modelled because they are mathematically simpler to deal with than the returns themselves.

Log returns (R) have been shown to approximate a normal distribution. In these cases, the returns (r) have what is known as a *log-normal* distribution. The log-normal distributions can be defined by three sets of parameters:

- the expected (log) return of each asset;
- the *standard deviation* of the (log) return of each asset; and

• the *correlations* between the (log) return of each pair of assets.

For the remainder of this chapter the reader should assume that when returns are being discussed it refers to log-returns.

In the example investor's case she is saving for her retirement; she is interested in maximising the purchasing power of her investment. Hence long-term, real after-tax returns are probably the relevant measure. As an example, (and without loss of generality) this chapter will assume that there are seven assets in which she may invest her wealth. The seven assets and their expected returns are shown in *Exhibit 12.5*. In this case they are forecast returns,[3] but they could be estimated from historical data.

Exhibit 12.5

Expected Asset Returns

Asset	**Expected annual, real after tax returns**
Equities (EQ)	5.00%
International Equities (IE)	4.00%
Listed Property (PR)	2.75%
Index Bonds (XB)	2.00%
Nominal Bonds (NB)	2.00%
International Bonds—hedged (IB)	1.50%
Cash (CH)	0.50%

The standard-deviation of an asset indicates how widely spread are the returns.

Exhibit 12.6 compares the distribution for nominal bonds and equities. The horizontal axis shows above/below average returns. Equities, being the more volatile asset class, have a much wider distribution of outcomes. Often the term "variance" is used in place of standard-deviation. Variance is simply the square of the standard-deviation.

3. This chapter was written from an Australian tax perspective. The effect of local tax laws such as tax credit received with income earned on domestic equity has been taken into account. The same assumptions would result in a different set of expected returns for non-Australian investors.

Exhibit 12.6

Standard Deviation of Equities and Bonds

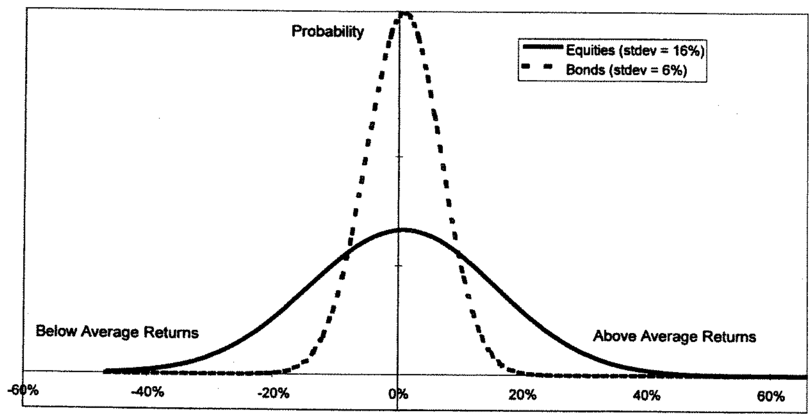


Exhibit 12.7 shows the annualised standard deviations for each asset class. These have been estimated from monthly total real returns over the period 1988 to 1995.

Exhibit 12.7

Standard Deviations of Assets

Asset	Annualised Standard Deviation
Equities (EQ)	15%
International Equities (IE)	16%
Property (PR)	9%
Index Bonds (XB)	6%
Nominal Bonds (NB)	5%
International Bonds—hedged (IB)	4%
Cash (CH)	1%

Correlations between two assets range between minus one and plus one. A negative/positive correlation indicates that when one asset has an above/below average return the other will tend to produce a below/above average return respectively.

Exhibit 12.8 gives a graphical representation of correlations. The dots show the actual monthly joint return for Australian equities and property over the period 1988 to 1995. The correlation calculated over this period is 0.55, a strong positive correlation. The ellipses show how the joint returns should be distributed assuming a log-normal distribution. For example, 10% of all observations should fall in the smallest ellipse. The effect of the high correlation between equities and property is to elongate the ellipses such that when equity returns are above/below average, property returns are more likely to be above/below average as well.

Exhibit 12.8
Correlation

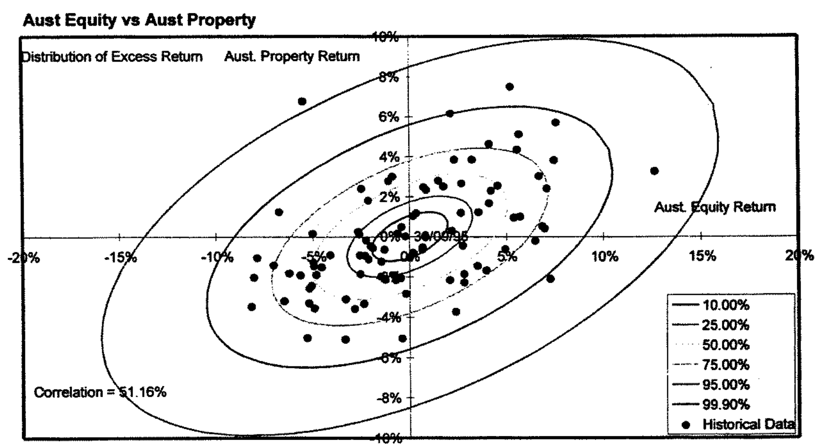


Exhibit 12.9 shows the correlation between the seven asset classes, estimated over the same period as the standard-deviations. Only half the matrix needs to be shown since the correlation of, for instance CH with IE, is the same as that of IE with CH (−2%).

Exhibit 12.9

Asset Correlations

	EQ	IE	PR	XB	NB	IB	CH
EQ	1.00						
IE	0.29	1.00					
PR	0.51	0.28	1.00				
XB	0.18	0.13	0.28	1.00			
NB	0.37	0.20	0.41	0.40	1.00		
IB	0.19	0.48	0.26	0.31	0.55	1.00	
CH	−0.01	−0.02	0.00	−0.14	0.19	−0.21	1.00

In statistics jargon the combination of the variances and correlations is called the co-variance matrix.

3. MEAN-VARIANCE OPTIMISATION

Armed with both a tractable model of investor preferences (the exponential utility function), and the probability distribution of returns (the log-normal distribution), the problem of selecting an optimal portfolio (one which is better than all other portfolios) can be solved.

The problem of portfolio optimisation is to maximise the expected value of utility given the expected distribution of returns. Mathematically this can be expressed as:

$$E[U(r)] = \int (1 - e^{-\lambda r})(2\pi\sigma)e^{\left(-\frac{1}{\sigma^2}\right)(r-\mu)(r-\mu)}dr$$

Surprisingly, this rather fearsome piece of mathematics can be simplified down to:[4]

$$\text{Maximise } \mu_p - \left(\frac{\lambda}{2}\right)\sigma_p^2$$

Where

μ_p is the expected return (mean) of the portfolio; and

σ^2_p is the expected variance of the portfolio.

This equation is known as the mean-variance criteria for an optimal portfolio. The criteria lends itself to a relatively simple interpretation. The optimal portfolio is the one that maximises the expected return of the portfolio, subject to a penalty for any increase in the expected variance. The higher the investor's risk aversion (λ) the greater the penalty for any variance in the portfolio return. Essentially, the variance of the portfolio becomes the

4. Refer to Appendix I for more details.

proxy for risk. For this reason, the co-variance matrix is often referred to as the risk matrix. For the remainder of this chapter the word "risk" will be used as a short-hand way of saying the standard-deviation of returns.

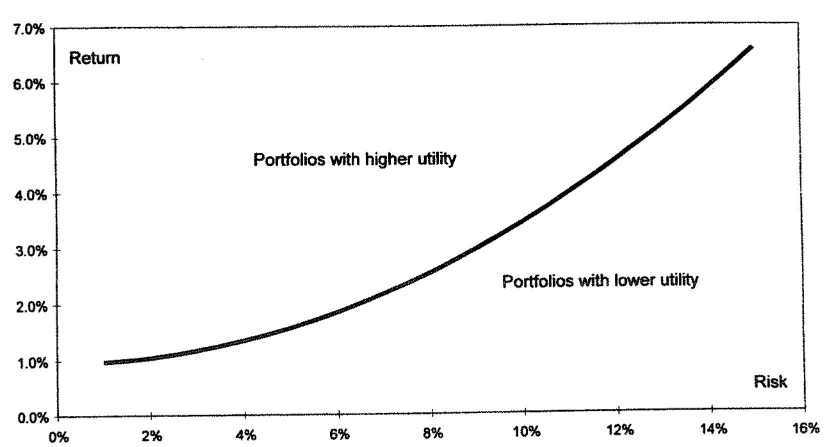
One of the major criticisms of the mean-variance model is that variance may not be a complete measure of risk. To an extent, this criticism is misguided as the use of variance as a measure of risk has been derived *indirectly* by assuming that returns have a log-normal distribution. Hence, the questioning of variance as a valid risk measure is really asking a deeper question of whether the log-normal distribution is a robust model of portfolio returns.

3.1 Indifference curves

The mean-variance criteria ($\mu_p - \lambda/2\sigma^2_p$) provides a way of comparing two portfolios. Suppose a portfolio is picked at random and its expected return and variance is calculated. Assume we have another portfolio with a higher return (say 1% higher) but with a higher variance (say $\lambda/2$% higher). The investor would be indifferent between the two portfolios. This is because the increase in utility due to the increase in the expected return is exactly offset by the decrease in utility due to the higher portfolio variance. If all portfolios with the same level of utility are joined, they form what is known as an indifference curve (IC). All portfolios on the IC are equally attractive to the investor. *Exhibit 12.10* shows a single IC. Most importantly, all portfolios that lie above/below the indifference curve are unambiguously better/worse than those on the IC.

Exhibit 12.10

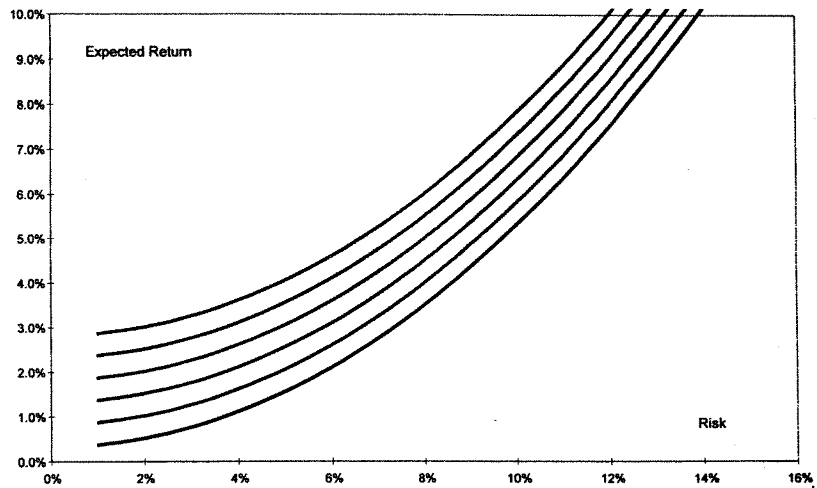
A Single Indifference Curve (λ = 10)



For each level of utility there will be an IC containing all portfolios that have that level of utility. *Exhibit 12.11* shows a range of utility curves.

Exhibit 12.11

A Range of Indifference Curves ($\lambda = 10$)



The investor's goal would to be on the highest possible IC. However, not all combinations of returns and variances are possible. For example, a portfolio with a very high return and no risk is highly desirable but, unfortunately, impossible to achieve. In order to determine the optimal portfolio we need to combine the IC with the range of portfolios that are feasible.

3.2 The efficient frontier

Using the values for the assets' expected returns, standard-deviation and correlations, the expected return and risk of any portfolio can be calculated. For example, *Exhibit 12.12* shows the expected return and risk of a hypothetical starting portfolio. Assume that this is the portfolio the investor is starting with before she has considered the problem of whether it is optimal.

Exhibit 12.12

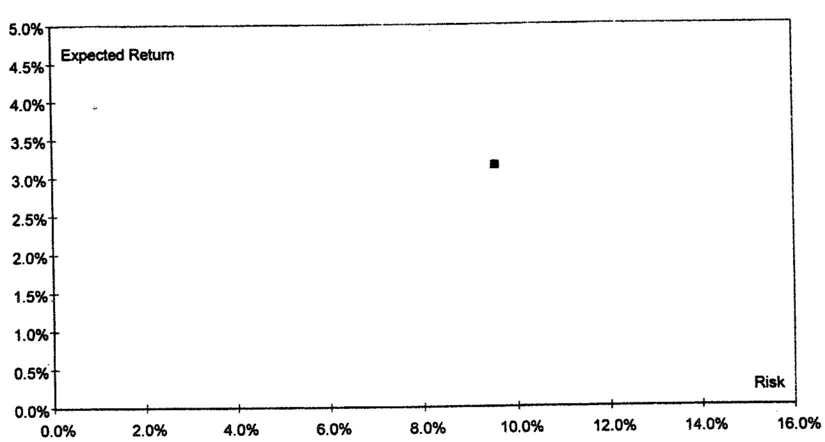
Expected Risk and Return of the Starting Portfolio

Asset	Weight in Starting Portfolio
Equities	2.7%
International Equities	54.3%
Property	2.5%
Index Bonds	3.7%
Nominal Bonds	33.5%
International Bonds—hedged	3.3%
Cash	0.0%
Total	100.0%
Expected return	3.2%
Risk	9.5%

The risk and return can be plotted on the same axes on which the indifference curves were plotted.

Exhibit 12.13

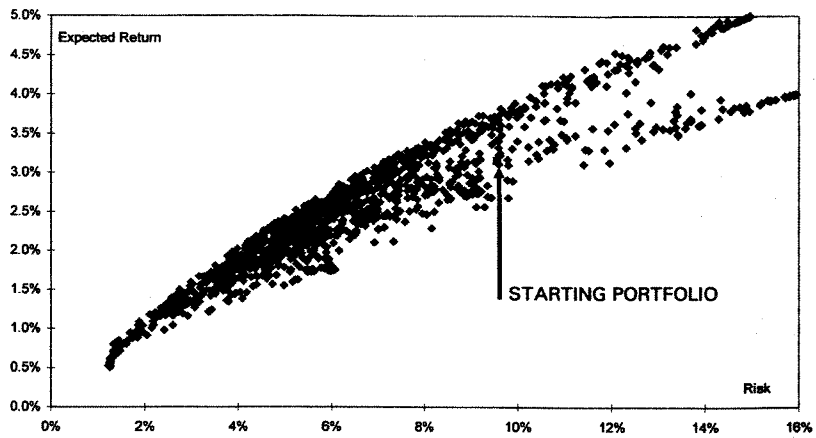
Risk and Expected Return of the Starting Portfolio



Being able to plot the risk and return characteristics of the starting portfolio is not very illuminating as it does not give an indication as to whether this portfolio is better or worse than other portfolios picked at random. A broader comparison is required. *Exhibit 12.14* shows the risk and return characteristics of 1000 portfolios selected at random along with the initial random portfolio.

Exhibit 12.14

Risk and Expected Return of a 1000 Random Portfolios

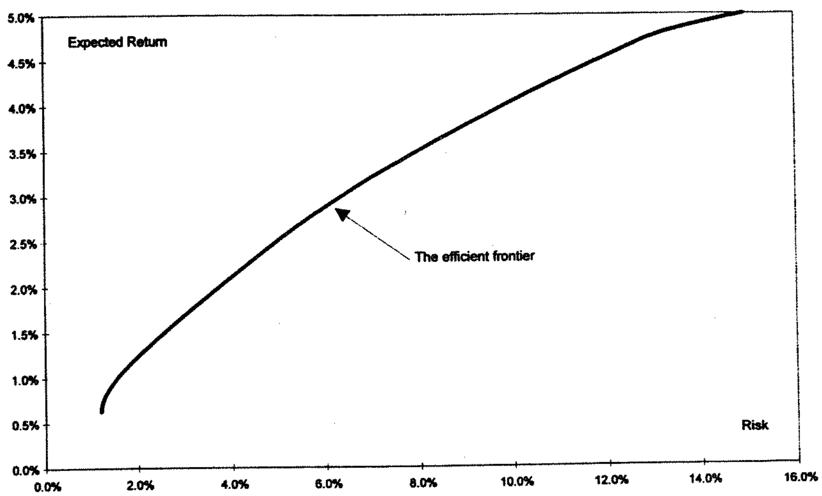


It is clear from this figure that the starting portfolio is not a candidate for being an optimal portfolio as there are many portfolios that have the same (or lower) levels of risk but a higher return. The starting portfolio is therefore said to be "sub-optimal" because there are portfolios which are unambiguously better. Note that the starting portfolio can be ruled out even before the risk aversion of the investor is considered. Once all the sub-optimal portfolios have been removed, the remaining portfolios each have the characteristic of the highest possible return for their level of risk. This set of portfolios is known by the slightly heroic name of "the efficient frontier". The "optimal" portfolio, for a given investor, will be somewhere along the efficient frontier.

Exhibit 12.15 shows the risk and return characteristics of the efficient frontier.

Exhibit 12.15

The Efficient Frontier



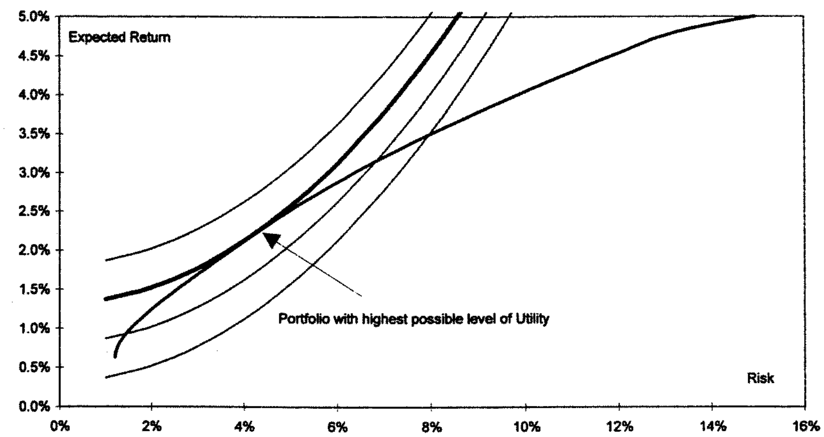
Fortunately, it is not necessary to create 1000 random portfolios to find the efficient frontier. A mathematical algorithm using *quadratic programming* can calculate the frontier. Further reading on the mechanics of calculating the efficient frontier can be found in the bibliography.

3.3 The optimal portfolio

The final question to answer is, given an investor's level of risk aversion, which portfolio on the efficient frontier is optimal? This can be answered by combining the efficient frontier with the investor's indifference curve. Recall that the investor's aim would be to have a portfolio that lies on the highest possible IC. *Exhibit 12.16* shows the efficient frontier along with the utility curves for an investor with $\lambda = 10$.

Exhibit 12.16

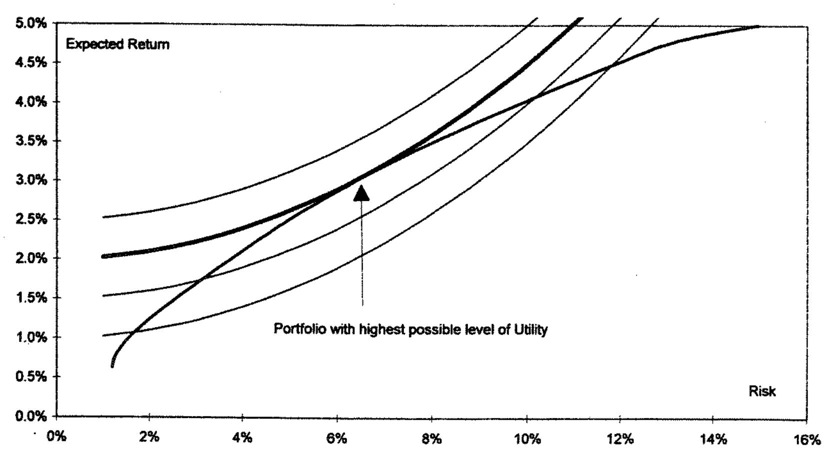
Optimal Portfolio for an Investor with λ = 10



The optimal portfolio is the point on the efficient frontier where the IC just touches. Higher IC are impossible to achieve and lower IC have a lower level of utility. The example investor has a λ of 5, with the optimal portfolio on the efficient frontier shown in *Exhibit 12.17.*

Exhibit 12.17

The Example Investor's Optimal Portfolio for λ = 5



Note the different shape of the IC due to the lower level of risk aversion. As the risk aversion decreases, portfolios with progressively higher returns (and risk) are selected. The weights of the portfolio are shown in *Exhibit 12.18* and the test portfolio is shown for comparison purposes.

Exhibit 12.18

Expected Risk and Return of the Optimal Portfolio $\lambda = 5$

Asset	Weight in Optimal Portfolio	Weight in Starting Portfolio
Equities	25.7%	2.7%
International Equities	13.5%	54.3%
Property	3.7%	2.5%
Index Bonds	33.4%	3.7%
Nominal Bonds	23.7%	33.5%
International Bonds—hedged	0.0%	3.3%
Cash	0.0%	0.0%
Total	100.0%	100.0%
Expected return	3.1%	3.2%
Risk (standard deviation of return)	6.6%	9.5%

Compared to the test portfolio the optimal portfolio has a significantly lower risk with only a slight fall in the expected return.

3.4 How reliable is the optimal portfolio?

If you saw a dice for the very first time and you knew nothing about the underlying probabilities of various numbers occurring on each roll, how could you determine the likelihood of each number occurring? One way would be to roll the dice 100 times and see how frequently each number comes up. The results of such an experiment are shown in *Exhibit 12.19*.

Exhibit 12.19

Frequency of Numbers in 100 Dice Rolls

Number	Frequency	Estimated Probability
1	17	0.17
2	9	0.09
3	22	0.22
4	13	0.13
5	18	0.18
6	21	0.21

The true probability of any number occurring is 0.1667 (1 in 6) but t estimated probabilities are significantly different from this. In the abo experiment the estimated chances of a two are less than 1 in 10 and t estimated chances of a six are better than 1 in 5. This type of inaccuracy called *estimation error*. If you were to base the strategy for playing dice-based game on these probabilities your approach would not be optimal

The optimal portfolio for the example investor seems to be qui reasonable. It has a good spread of assets and, *based on the estimates f return and risk*, it is has a higher utility than any other portfoli Unfortunately, the estimates of risk and return are just that—estimates. with the dice, actual underlying risks and returns are still unknowns. Th can only be estimated either by some forecasting method and/or observati of historical behaviour. An important question to explore is how much do estimation error affect the optimal portfolio?

The simple answer is that it affects the portfolio quite substantially. F example, consider the following experiment. Here the optimal portfolio h been calculated 100 times—each time with a slightly different co-varian matrix, in order to simulate the effects of sampling error.[5] *Exhibit 12.2* shows the range of outcome for each asset class. *Exhibit 12.21* shows tl actual weights for the 100 simulated portfolios.

5. The additional co-variance matrices were calculated using a bootstrap method. Rando samples of the multi-variate normal distribution were created based on the original co-varian matrix. From these samples the new co-variance matrix was estimated.

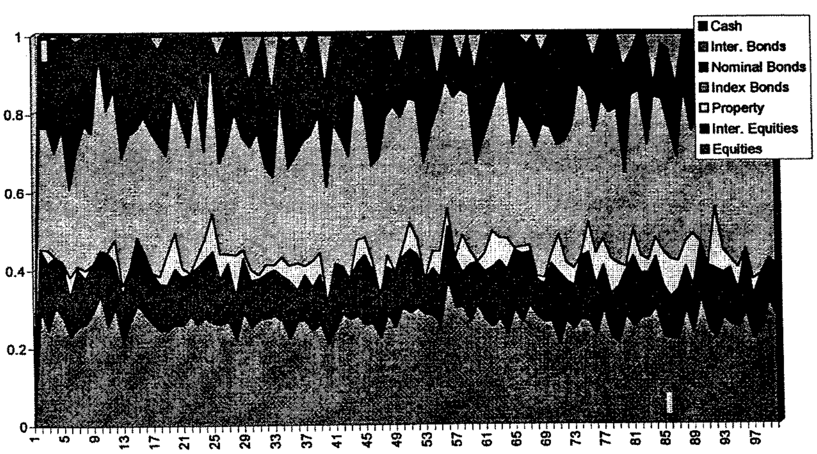
Exhibit 12.20

Range of Optimal Asset Weights in Sampling Error Experiment

Asset	Minimum Weight	Maximum Weight	Average Weight
Equities	20.5%	37.2%	26.6%
International Equities	8.0%	19.1%	13.3%
Property	0.0%	15.7%	3.7%
Index Bonds	22.5%	49.4%	33.3%
Nominal Bonds	3.4%	39.0%	21.5%
International Bonds—hedged	0.0%	14.4%	1.6%
Cash	0.0%	0.0%	0.0%

Exhibit 12.21

Range of Optimal Asset Weights in Sampling Error Experiment



The average weights are close to the original optimal portfolio but the range that the optimal weights can take is quite startling. Take, for example, the optimal weight for Australian equities which varies between 20.5% and 37.2%. Also, the optimiser has not taken into account such features as transaction costs and the lack of liquidity in some asset classes. Note that none of the underlying assumptions of the model (such as returns having a log-normal distribution) has been questioned in this experiment.

The sensitivity and lack of reality of the mean-variance optimised portfolios are well known to practitioners and are the primary reasons why, despite the appealing theory, the mean-variance optimisers are not widely used in portfolio construction. In a chapter entitled portfolio optimisation this is a rather disappointing observation. The next section investigates methods by which the mean-variance model can be tamed.

4. PRACTICAL PORTFOLIO OPTIMISATION

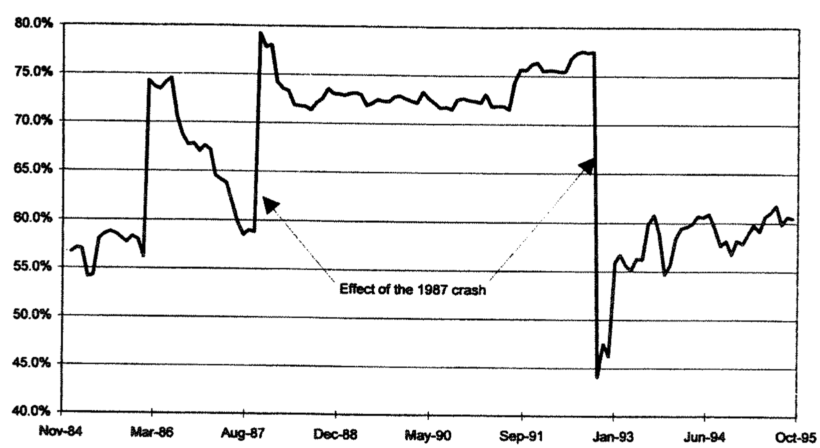
When the mean-variance model is given a set of expected returns and a co-variance matrix *and nothing* else it is not surprising that the output is sensitive to these inputs and lacking in realism to such issues as liquidity and transaction costs. It was not told anything about them. Broadly, the mean-variance model can be improved in two ways. First, by stabilising the variability in the inputs and, secondly, by expanding the model to include other information.

4.1 Robust risk estimation

Exhibit 12.22 shows the rolling correlation between Australian equity and property measured, using five years of monthly returns. There are a number of instances of the correlation jumping substantially in a single month, notably just after the market crash in October 1987. The correlation also changes significantly in October 1992 when the 1987 crash drops out of the sample period.

Exhibit 12.22

Rolling 5 year Correlation Between Equity and Property



It could be argued (tenuously) that the rise in correlation in October 1987 was real, but it is very difficult to support the idea that the correlation between equity and property fell by 50% in October 1992 just because it was 5 years and 1 month after the crash. In the previous section it was noted that the output from a mean-variance model can vary significantly with change in the risk measures. It can be rather disturbing to an analyst to find the optimiser suggesting a substantially different portfolio when there have been no significant changes to the forecast returns.

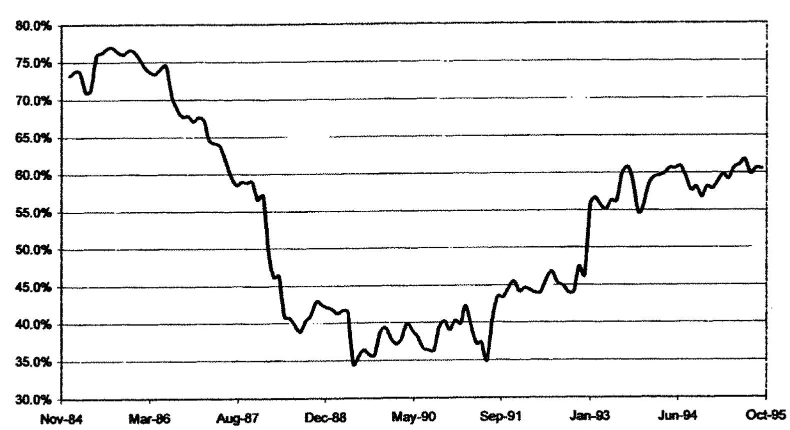
In choosing the methods to calculate the co-variance matrix, there should be two goals:

- ensure that the risk measure does not change dramatically from one period to another when there has been no significant change in the market; and
- ensure that the co-variance matrix is free from extreme values that unduly affect the optimisation process.

This is known as robust estimation. A detailed description of the methods for robustly estimating a co-variance matrix are beyond the scope of this book. To illustrate the idea, a simple approach, that can dramatically improve the stability of the risk measures, is to exclude extreme market moves (such as October 1987). *Exhibit 12.23* shows the correlation between equity and property over the same period after the two most extreme months have been removed.

Exhibit 12.23

Robust Rolling Correlation Between Equity and Property



The advantage of using robust techniques to estimate the co-variance matrix is that it will allow the mean-variance model to produce consistent results over time, and limit the extent to which output will be greeted with the response "why is it doing that?" The co-variance matrix will still be subject to estimation error but, because it is more stable over time, analysts can learn to work with the model.

4.2 Optimisation to a benchmark and asset/liability modelling

The two-asset problem, introduced earlier, is good for illustrating the idea of risk aversion but in practice a more detailed analysis of the investor's requirements is needed to determine their level of risk aversion.

For example, consider a young man with at least thirty years before retirement. For such a person the "safe" investment A could well represent an extremely risky prospect. Assume that the basket of goods the young man is aiming to consume[6] was increasing in cost at a rate of 4% per annum over the next 30 years. In this case investment A would result in a slow but steady erosion of the future purchasing power of his nest egg. The young man would be certain of the amount of money he would have, but uncertain if he could afford anything. His best course of action would be to invest substantially in the "risky" investment B.

This type of analysis is known as asset/liability modelling. In this case the liability is the basket of goods to be consumed in retirement in 30 years time. The aim of asset/liability modelling is to create a long-term portfolio which, as closely as possible, matches the liability profile of the investor.

Such a portfolio can be used as the starting point for a mean-variance model. When used in this way the long-term portfolio is called a benchmark, strategic or a normal portfolio (is this chapter it will be referred to as a benchmark). The benchmark embodies the long-term assumptions about the behaviour of the investor's liabilities and the assets in which he or she can invest.

Asset/liability modelling is not the only source of benchmarks. Frequently, fund managers are given an implicit benchmark of the average allocation of their peers. Specialist managers (those managing one asset class) typically are given the broad market index as their benchmark.

The use of a long-term benchmark transforms the mean-variance problem in a number of ways. First, risk is defined in terms of variance in returns relative to the benchmark return. This variance is often called *tracking error*. Secondly, the expected returns may need to be expressed in terms of deviations from the long-term returns. For example, if the forecast for Australian equities for the next year was 4% this is 1% below the long-term expected return (assuming the benchmark has been constructed using returns from *Exhibit 12.5*). Hence, the relevant expected return for the optimiser is −1%. The reason for this is that the long-term expectation of Australian equities earning an after-tax return of 5% is already embodied in the benchmark. If the long-term return was included in the optimisation relative to the benchmark it would be double counted. *Exhibit 12.24* shows an example long-term benchmark and forecast asset returns.

6. The basket may consist of such goods as a house by the beach, health care, luxury motor cars and overseas trips. The CPI may be a poor indicator of rate of change in the price of this basket.

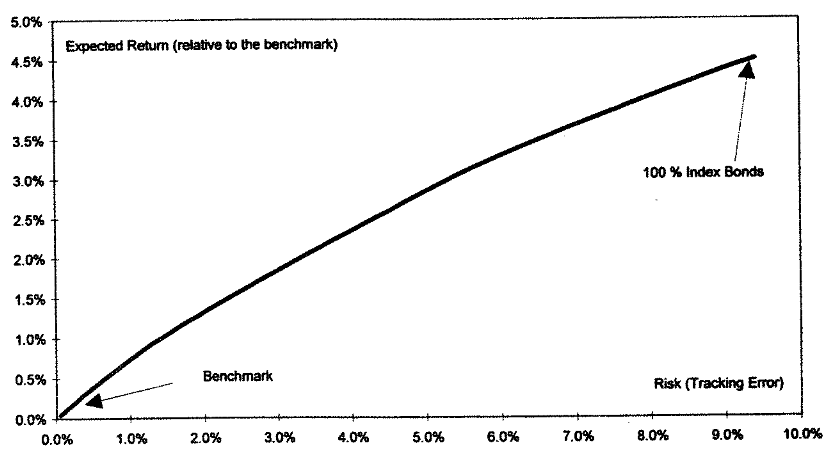
Exhibit 12.24

Benchmark and Forecast Expected Returns

Asset	Forecast Expected Returns (relative to long-term returns)	Benchmark Weight (%)
Equities	−2%	40.0
International Equities	−5%	25.0
Property	−3%	10.0
Index Bonds	2.5%	5.0
Nominal Bonds	1.5%	15.0
International Bonds—hedged	1%	0.0
Cash	0%	5.0
Expected return		−2.0%

Exhibit 12.25

Efficient Frontier with a Benchmark Portfolio



Optimising to a benchmark produces significantly more stable results than optimising the absolute portfolio weights. The starting point for the efficient frontier will *always* be the benchmark, regardless of the forecast returns and estimated co-variance matrix.

Exhibit 12.26

Selected Portfolios on the Efficient Frontier

Asset	Lowest Risk (Benchmark) (%)	Medium Risk (%)	Highest Risk (%)
Equities	40.0	36.4	
International Equities	25.0	13.5	
Property	10.0	0.0	
Index Bonds	5.0	25.3	100.0
Nominal Bonds	15.0	24.8	
International Bonds—hedged	0.0	0.0	
Cash	5.0	0.0	
Risk (tracking error)	0.0	2.5	9.4
Expected return	−2.0	0.4	2.5

Note that returns in this table returns are relative to the benchmark.

4.3 Transaction costs, turnover constraints and asset bounds

In *Exhibit 12.26* the portfolios on the efficient frontier contained significant amounts of index bonds. This is understandable as index bonds have the highest expected return. However, index bonds are not very liquid and consequently the costs of investing (and divesting at a later date) are high. This has not been taken into account in the optimiser.

The mean-variance model can easily be extended to include the cost of transactions. This is achieved by effectively reducing the expected return of a portfolio by the costs involved in trading from the existing portfolio to the new one. A further refinement is to place differential transaction costs depending on whether the portfolio weight is being moved towards or away from the benchmark. The costs of moving away from the benchmark are double the actual cost, whereas the costs of moving towards the benchmark are zero. The logic of this is that the portfolio will only move away from the benchmark if the expected return can cover the round trip costs of investment.

The optimal portfolio can be further constrained by limiting overall turnover of the portfolio. For example, it could be decided that the total turnover should be no more than some pre-set limit, for instance 20%.

The bluntest technique for controlling portfolio turnover is to apply bounds to the asset classes. For example, it could be decided independently of the optimisation process that the maximum weight in which the portfolio could realistically be invested in index bonds is 10%. Ideally, if the expected returns, transactions costs and overall risk aversion have being properly specified then explicit bounds on assets should not be required. Overuse of asset bounds can lead to the optimal portfolio being primarily defined by the bounds rather than by the optimisation process.

On the other hand, fund managers are often given explicit guidelines in terms of how much they can invest in a particular asset. Bounds play a very useful role when they represent such pre-defined (and often legally binding) restrictions on a fund.

4.4 Bayesian Inference—allowing for imperfect forecasts

So far, very little has been said about the estimated expected returns. They are the most important input of the model. A model with a poorly constructed or unstable co-variance matrix could still produce a portfolio that will add value to the benchmark—*if the expected returns are accurate*. If the expected returns are poor, the resultant portfolio will almost certainly under-perform its benchmark.

One of the most persistent criticisms of the mean-variance model from an investment analyst's point is that there is no allowance for some forecasts to be better than others. For example, an analyst asked to forecast equities and property might say:

> "Well, I'm fairly confident the equities will kick on and return 10%. On property—I guess it might ease a bit—but in this market it's difficult to call."

The exasperated person in charge of running the optimiser will press the analyst to give an unambiguous, single forecast for each asset. Assume, in this case, the forecasts end up being equities up 10% and property down 2%. Is this the best interpretation of the analyst's views? There are two elements missing from this final forecast. First, the analyst is clearly uncertain of the property forecast. The mean-variance model would treat these two forecasts as being equally valid. Secondly, there is a known strong positive correlation between equity and property which is not necessarily being taken into account.

The problem of combining imperfect knowledge with sample information is an area of statistics known as Bayesian Inference. An approach suggested by Black and Litterman is particularly novel and innovative. It notes that there are two competing sources of returns, the underlying long-run returns and the analyst's view. Both sources of information are imperfect and best expressed in probabilistic terms. Also, the co-variance matrix gives information about the pattern of joint returns.

Instead of pressing that analyst to give an expected return for property that has an equal level of conviction, the analyst could be allowed to specify the level of confidence with which each view is held. *Exhibit 12.27* shows the analyst forecasts together with the assumed long-term returns[7] and confidence level of each forecast.

7. If the optimisation was relative to a benchmark then the long-term returns would be zero, as they have already been embodied in the benchmark.

Exhibit 12.27

Long-term Return, Analyst's Forecasts and Confidence Levels

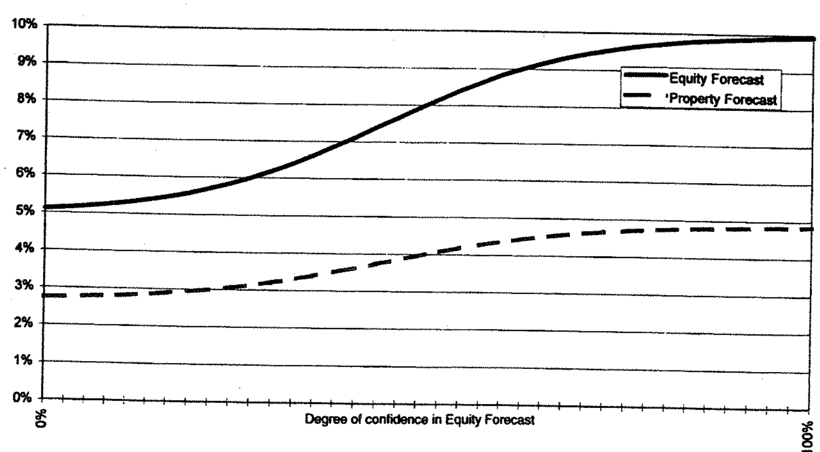
Asset class	Long-term return	Analyst's forecast	Confidence level
Equities	5.00%	10%	?
Property	2.75%	−2%	5%

In the above table the level of confidence for the equity forecast has not been specified.

Exhibit 12.28 shows how the forecasts for both equity and property vary as the confidence in the equity forecast grows.[8] When there is no confidence, both the equity and property forecasts are at their long-run values. As the level of confidence increases, the equity forecast moves toward the analyst's forecast. Interestingly, the property forecast rises, *away from the analyst's forecast*. Equities are strongly correlated with the property market, as the confidence in the analyst's forecast grows, it implies that property will also rise. This indirect influence has a greater impact because the direct view on property is held so weakly.

Exhibit 12.28

Forecast Returns with Differing Levels of Confidence using the Black-Litterman Model



8. Refer to Appendix II for more details of the Black-Litterman model.

Although this two asset example is unrealistic, the approach becomes much more valuable as the number of assets increases. Consider a group of analysts with the task of analysing a few hundred stocks to create an equity portfolio. It is unreasonable for every stock to be analysed and forecast with the same level of detail. The Black-Litterman model allows for a varied level of analysis to be performed, even accommodating stocks on which there has been no analysis.

5. OTHER OPTIMISATION APPROACHES

This section will explore some other approaches to creating an ''optimal'' portfolio.

5.1 Scenario-based optimisation (goal programming)

As previously noted, one criticism from practitioners of the mean-variance model is that it requires single estimates of the mean return for each asset. Investment analysts often think in terms of scenarios. Scenario-based optimisation allows for forecasts to be input in terms of a number of sets of forecasts (usually at least three), each with an associated probability. *Exhibit 12.29* shows an example of three forecast scenarios along with their associated probability. *Exhibit 12.30* shows an example benchmark and some lower and upper bounds for a portfolio.

Exhibit 12.29

Forecast Real Returns for Three Scenarios

	"Boom" Scenario	"Normal" Scenario	"Bust" Scenario
Probability	15%	70%	15%
Equities	40.0%	5.0%	−20.0%
International Equities	25.0%	4.5%	−25.0%
Property	3.5%	3.5%	3.5%
Index Bonds	7.0%	3.0%	−10.0%
Nominal Bonds	0.0%	2.5%	−10.0%
International bonds—hedged	0.0%	1.0%	−8.0%
Cash	1.5%	1.0%	−0.5%

Exhibit 12.30

Portfolio's Benchmark, Lower and Upper Bounds

	Lower Bound	Benchmark	Upper Bound
Equities	20%	40.0%	60%
International Equities	15%	25.0%	35%
Property	5%	10.0%	15%
Index Bonds	0%	5.0%	10%
Nominal Bonds	5%	15.0%	25%
International Bonds—hedged	0%	0.0%	10%
Cash	0%	5.0%	20%

A typical example of scenario optimisation is to find the portfolio with the highest expected (probability weighted) return subject to the constraint that the portfolio does not under-perform the benchmark in any single scenario. *Exhibit 12.31* shows the resultant optimal portfolio. *Exhibit 12.32* shows the return of the portfolio and benchmark under each scenario.

Exhibit 12.31

Portfolio with the Highest Expected Return (Subject to not Under-performing the Benchmark)

	Optimal Weights
Equities	60.0%
International Equities	15.7%
Property	15.0%
Index Bonds	0.0%
Nominal Bonds	5.0%
International Bonds—hedged	0.0%
Cash	4.3%

Exhibit 12.32

Returns of Portfolio and Benchmark

	Benchmark	Portfolio	Relative
Normal	4.1%	4.4%	0.4%
Boom	23.0%	28.5%	5.5%
Bust	−15.9%	−15.9%	0.0%
Expected	3.9%	5.0%	1.1%

The difference in the expected return of the portfolio and benchmark is 1.1% and there are no scenarios where the portfolio under-performs the benchmark. So it would appear that the scenario optimisation has been successful.

However, there are some drawbacks with this type of approach. First, the portfolio is largely at the extreme end of the asset ranges. This is known in optimisation as a *corner solution* where the result lies at the boundaries of the problem. The reasons for this are subtle. If *Exhibit 12.29* is studied closely, one can note that in all scenarios the local equities market is forecast to outperform the international equity market. Hence, it is possible to over-weight local equities and under-weight international equities ad infinitum, all the time increasing the expected returns without under-performing in any scenario.

The underlying problem is in using a limited number of scenarios to describe all possible futures. This problem is not easily solved merely by taking care not to produce forecasts with embedded free lunches, as there can be groups of assets with the same problem. For example, in the above set of forecasts, a combination of index bond and international bonds can be constructed that will always out-perform nominal bonds.

The problem of scenario optimisers tending to produce extreme solutions can be mitigated by adding addition constraints (turnover, transaction cost) or penalising the portfolio for an increase in risk. However this needs to be thoughtfully done or it can be unclear as to whether the resultant portfolio is a product of the forecasts or of the choice of "taming" constraints.

Scenario-based approaches are most applicable where the scenarios do describe the complete range of outcomes.

5.2 Generalised (non-parametric) portfolio optimisation

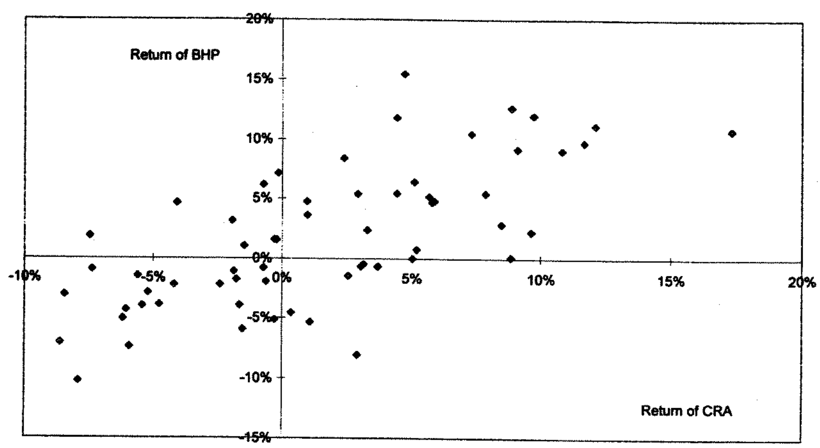
The simple and elegant mean-variance criteria is derived by assuming that return are normally distributed. Hence, disagreement with variance as a risk measure is largely an indirect disagreement with the assumption of normality. In the case of some derivative securities (such as options) the assumption of normality is quite clearly wrong.

An approach that avoids assuming any particular form for the distribution of returns is empirical or non-parametric optimisation. Rather than using return observations (be they historical, from a model, or forecast) to estimate a co-variance matrix and expected returns, the returns are instead used to directly describe the probability distribution. This idea is best explained by way of an example.

Exhibit 12.33 shows the monthly historical returns of two Australian stocks (BHP and CRA) over the period 1990 to 1995. The joint distribution is reasonably well described by the stock means and variances and their co-variance.

Exhibit 12.33

Joint Returns of BHP and CRA



Consider a third stock which has some option characteristics. Assume the stock will either return the highest return of BHP and CRA, or expire worthless if both stocks have a negative return. Further, assume for this example that this option costs 3.5% in each period. *Exhibit 12.34* and *Exhibit 12.35* show the joint distribution of the option with BHP and CRA stocks respectively.

Exhibit 12.34

Joint Returns of BHP and Hypothetical Option

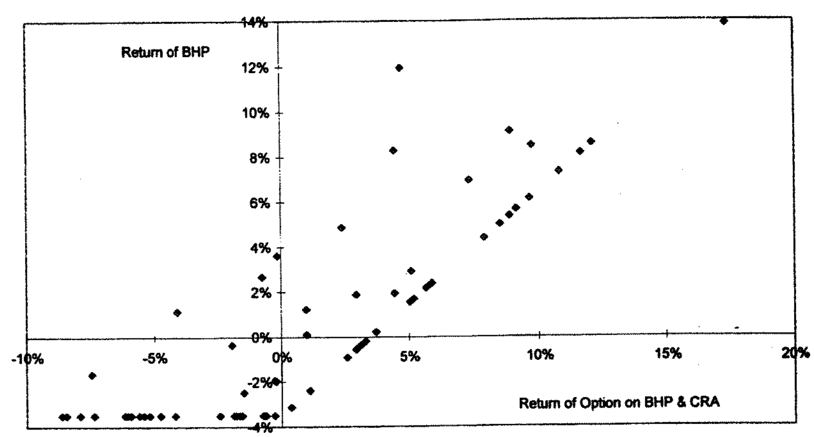
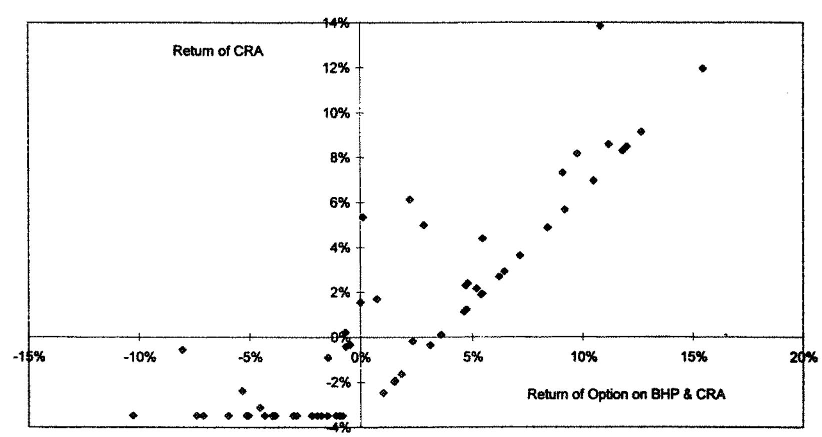


Exhibit 12.35

Joint Returns of CRA and Hypothetical Option



In both cases, using mean, variance and co-variance to describe the joint distribution removes a great deal of information, specifically the asymmetric behaviour of the option. An empirical approach does not try to summarise the entire distribution in a few parameters. Rather, each observation is used directly to represent the future possible outcomes. The most naive approach is to assume that each historical outcome is equally likely to occur in the future. The portfolio optimisation problem then becomes to maximise utility over the empirical distribution.

Exhibit 12.36 shows the summary statistics of the three assets that would be the input into a mean-variance optimiser. The option has a significantly lower expected return and a slightly lower risk. *Exhibit 12.37* shows the resultant optimal mix of BHP and CRA for a reasonably risk-averse investor (λ = 12) for both the mean-variance approach and the empirical approach. *Exhibit 12.38* shows the optimal portfolio derived from the two approaches when the investor is able to purchase the option stock.

Exhibit 12.36

Return, Risk and Correlation of BHP, CRA and Option

	BHP	CRA	Option
Expected Return	18.3%	20.5%	10.5%
Risk (Standard deviation)	20.4%	20.4%	15.9%
Correlation with CRA	70%	------	------
Correlation with option	85%	88%	------

Exhibit 12.37

Optimal Portfolios for Mean-variance and Empirical Optimisation (λ = 12)—Option not Included

	Mean-variance	Empirical
BHP	42.4%	40.5%
CRA	57.5%	59.5%

Exhibit 12.38

Optimal Portfolios for Mean-Variance and Empirical Optimisation ($\lambda = 12$)—Option Included

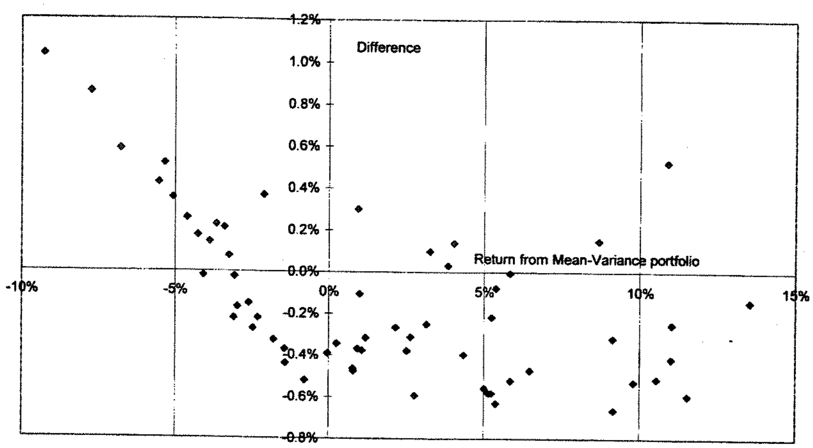
	Mean-variance	Empirical
BHP	42.4%	31.3%
CRA	57.5%	49.5%
Option	0.0%	19.1%

For a risk-averse investor the option is an attractive investment, as it enables the construction of a portfolio with a more limited down-side. As the mean-variance model assumes that returns are symmetrically distributed, it is unable to "see" the asymmetric nature of the option, and only reviews its risk characteristics based on its variance and co-variance with the other assets. The empirical approach uses the actual historical outcomes to characterise its expectation of future behaviour, and hence it is able to "see" the value in the option.

This point is illustrated in *Exhibit 12.39*. On the horizontal axis is the return of the mean-variance optimal portfolio if it had been invested over the sample period from 1990 to 1995. The vertical axis shows the difference in return between the empirical portfolio and the mean-variance portfolio. The empirical portfolio out-performs the mean variance portfolio significantly when its returns are negative and largely matches or marginally under-performs it at other times. Essentially, the empirical portfolio has traded-off a slightly lower expected return for a significantly lower risk of very poor returns. To a risk-averse investor, this is a good deal.

Exhibit 12.39

Difference in Returns Between Empirical and Mean-variance Optimisation



Given the slightly magical ability of a non-parametric optimisation approach to identify and exploit asymmetric distributions, it is tempting to think that it should always be used in preference to a mean-variance model. However, if the underlying distribution of returns are actually (or approximately) normal then the non-parametric approach will magnify the sampling error effect. Also, the non-parametric approach does not lead to a clear definition of risk, which can make interpretation of results more difficult.

6. CHAPTER SUMMARY

The mean-variance model for calculating an optimal portfolio makes two key assumptions:

- investors' preferences can be expressed in terms of well behaved and exponential utility function; and
- returns are modelled with a log-normal distribution.

This leads to the mean-variance criteria for an optimal portfolio (Maximise $\mu_p - \lambda/2\sigma^2{}_p$). This can be interpreted to mean that an investor will only accept a portfolio with a higher risk provided there is a commensurately higher return. The risk-aversion parameter (λ) can be thought of as the investor's pricing of risk.

Despite its elegant simplicity the mean-variance model has been criticised by practitioners because the output is highly sensitive to small changes in either the expected returns or co-variance matrix. Also the model can produce results that are unrealistic when market factors such as liquidity and transaction costs are considered. There are a variety of methods to overcome these problems, notably:

- using robust methods to calculate the co-variance matrix to reduce unrealistic changes and extreme values in the risk measures;
- using a benchmark as the starting point for an optimisation;
- including transaction costs and turnover limits;
- including legal and practical restrictions in the form of bounds; and
- modifying expected returns to take account of their imperfect nature and to allow for some views to be less firmly held than others.

Successful portfolio optimisation requires more than plugging risk and returns into a mean-variance optimiser and turning the handle. Care needs to be taken to ensure that the inputs are realistic, and that the model is expanded to cover the realities of the financial markets. Used appropriately, the mean-variance model is a powerful tool for efficiently translating estimates of risks and returns into portfolios.

Appendix I

Derivation of the Criteria for an Optimal Portfolio

A rational investor with an exponential utility function will seek to maximise the expected utility. This is defined as:

$$\text{Maximise } E[U(r)] = \int (1 - e^{-\gamma r})f(r)dr$$

or more simply:

$$\text{Minimise } E[U(r)] = \int e^{-\gamma r}f(r)dr$$

The form of this equation is well known to statisticians as the moment generating function (MGF) evaluated at $-\lambda$. Maximising the exponential utility function over a given distribution is equivalent to minimising the MGF of the distribution. The MGF for the normal distribution is:

$$MGF_{normal}(-\lambda) = e^{\left(-\lambda\mu + \frac{(-\lambda)^2\sigma^2}{2}\right)}$$

The above equation is minimised when the exponent value is minimised, that is

$$\text{Minimise } -\lambda\mu + \frac{\lambda^2\sigma^2}{2}$$

Dividing by $-\lambda$ gives:

$$\text{Maximise } \mu - \frac{\lambda}{2}\sigma^2$$

This is the now familiar mean-variance criteria for an optimal portfolio. An important point is that if the return distribution has a MGF in a closed form *then criteria for an optimal portfolio can be expressed directly.* For example, if the returns were uniformly distributed over the interval $[\theta_1, \theta_2]$ then the criteria for the optimal portfolio would be:

$$\text{Minimise } MGF_{uniform}(-\lambda) = \frac{e^{-\lambda\theta_2} - e^{-\lambda\theta_1}}{-\lambda(\theta_2 - \theta_1)}$$

Similarly, for a gamma distribution (expressed as a function of α and β, the criteria is:

$$\text{Minimise } MGF_{gamma}(-\lambda) = (1 + \beta\lambda)^{-a}.$$

Given the ease by which the criteria for an optimal portfolio can be expressed, why is it that the mean-variance criteria (derived from assuming a log normal distribution of returns) dominates finance, and other criteria are rarely seen?

There are two main reasons for this:

- the log normal distribution is, by and large, the best at describing the observed distribution of returns; and

- only the normal distribution can be generalised into a multi-variate problem without becoming overly complex mathematically.

However, this technique for deriving the criteria for an optimal portfolio is useful to apply to specific problems. The next Appendix gives an example of this.

Appendix II

Implied Risk Aversion from Two Asset Problem

An investor has a choice of two assets to invest in. One will provide a certain return of 3%, the other returning 40%, 10% and −20% with probabilities of 0.25, 0.50 and 0.25 respectively. If it is assumed the investor has an exponential utility function what can be inferred about the investors risk aversion based on the chosen weight in the risky asset?

The exponential utility function takes the following form:

$$U(r) = 1 - e^{-\lambda r}$$

where

λ is a risk-aversion parameter (pronounced lambda);

U is the level of utility for a given level of return; and

r is the return.

A rational investor will seek to maximise the expected utility. This is:

$$\text{Maximise } E[U(r)] = \int (1 - e^{-\lambda r}) f(r)dr$$

where $f(r)$ is the probability density function of r. In the above case the expected value of utility for a given weight (w) in the risky asset is:

$$E[U(r)] = 1 - \left(0.25e^{-\lambda(1+0.4w+(1-w)0.03)} + 0.50e^{-\lambda(1+0.1w+(1-w)0.03)} + 0.25e^{-\lambda(1-0.2w+(1-w)0.03)}\right)$$

The investor will seek to choose such that the expected utility is maximised. That is:

$$\frac{dE[u]}{dw} = 0$$

$$\frac{dE[u]}{dw} = \left(\frac{-\lambda 0.25}{100}\right)\left(37e^{-\lambda(0.97+0.37w)} + 14e^{-\lambda(0.97+0.07w)} - 23e^{-\lambda(0.97-0.23w)}\right) = 0$$

Dividing the above equation through by $\left(\frac{-\lambda 0.25}{100}\right)\left(\frac{e^{\lambda 0.97}}{e^{\lambda 0.23w}}\right)$ gives:

$$37e^{(-\lambda 0.6w)} + 14e^{(-\lambda 0.3w)} - 23 = 0$$

Let $x = e^{-\lambda 0.3w}$ and substitute in the above equation gives:

$$37x^2 + 14x - 23 = 0$$

Solving this quadratic gives a value for of either −1 or 23/37. As x cannot be negative it can be concluded that:

$$\lambda = \frac{-\ln(23/37)/0.3}{w} \cong \frac{1.58}{w}$$

This gives a simple relationship between the risky asset weight and the investor level of risk aversion.

Appendix III

The Black-Litterman Transformation

The expected return $E[r]$ in the Black-Litterman model is calculated as follows:

$$E[r] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right]$$

Where

- n is the number of assets.
- k is the number of views held by the investors. Note the k does not necessarily have to equal n as there may be assets on which the investor does not have a view.
- Σ is the n by n co-variance matrix.
- Q is the k by 1 vector of investor views.
- P is a k by n matrix indicating those assets for which an investor has expressed a view. Z will contain a one where a view has been expressed, and will be zero otherwise. Note that a relative view (excess return of an asset over another) can be expressed by a combination of one and minus one.
- Ω is a k by k diagonal matrix showing the uncertainty with which each view is held. The higher the i^{th} *diagonal value the more uncertain is the* i^{th} *view. It is assumed that* $P'E[r] = Q + \xi$, where ξ is a vector of normal variates with zero means. Ω is the co-variance matrix of ξ. Hence, as the variance of ξ approaches zero, $E[r]$ approaches the investor views.
- Π is a n by 1 vector of long-term equilibrium returns.
- τ is a scaler ($0 \geq \tau \geq 1$) that reflects the degree of uncertainty in the long term equilibrium returns. Zero would indicate that there is no uncertainty in the long-term equilibrium returns (and the investor views would be redundant).

Chapter 13

Risk Management for Bond Portfolios

by Roger Cohen

1. INTRODUCTION

Bond portfolio management is a major area in the financial markets. Some of the basic principles will be covered here. For further detail, the reader should consult references that specifically deal with this area.

The major areas of portfolio management are asset and liability management. The former applies to managers who purchase, hold and trade assets. Their main aim is to maximise return within specified constraints. Liability managers are trying to minimise their cost of funding by issuing appropriate instruments. They need to be able to service interest payments, to repay principal as scheduled, and to meet cash outflows when required. At times, the liability manager may find it advantageous to arrange refinancing, or to alter the mix of liabilities.

Although asset and liability management are complementary (the liability manager is trying to reduce cost, the asset manager is maximising return), the principles governing both sides are the same. The risks are the same, as are the techniques for managing them.

The following chapter is written mainly from the asset perspective. Implicit in this is the aim to maximise return, and to minimise risk. For liability management, the same techniques are applied to minimise cost.

2. PORTFOLIO MANAGEMENT

2.1 Matching and hedging

Matching and hedging are probably the fundamental concepts required in the management of any portfolio. They allow the portfolio manager to offset both risk and obligation in the portfolio. The former is done by removing sensitivity to possible external changes that may occur. These include changes in market rates, credit and other factors which affect the value of the portfolio.

Matching requires ensuring that a portfolio meets all cashflow obligations set on it. This means that cash inflows and outflows can be met without disrupting the portfolio. A perfectly matched portfolio will deliver exactly the cashflow required when it is needed. Cash inflows will be invested with this criteria in mind. There should also be provisions for emergencies by having a portion of the assets in highly liquid securities, or cash.

A perfectly hedged portfolio should contain no sensitivity to market factors (technically this is impossible, but within normal operating circumstances risk should be minimal). If for example, the portfolio value is going to decline drastically when rates increase, then any hedging strategy will need to hold assets (or liabilities) whose behaviour offsets this decrease.

The function of a portfolio manager is to supply the optimum amount of matching and hedging, and the appropriate choice of assets, so that the portfolio performs to, or exceeds, its goals. This has to be done within set constraints. These can be highly varied. They can govern the types of assets held, the maturity profile, duration limits and allowable trading. At the extremes, consider a long-term retirement portfolio versus an arbitrage trading portfolio. The retirement portfolio would probably hold quality assets with steady yield. The arbitrage portfolio will be traded actively for short-term gain. Style of portfolio management can be as varied as the purpose of the portfolio, and the aims of the manager. There is no perfect management method—rather a number of strategies which can be chosen or adapted. Portfolio management is in many ways a very individual discipline.

Of the many strategies available to the manager of a bond (or indeed other) portfolio, there are two broad categories into which most fall. These are the choice of active versus passive management. The former is used where the portfolio is measured and adjusted over very short time periods. Often this occurs many times a day. The aim is to control risk while taking advantage of intraday market movements. The composition of the portfolio can change rapidly. Passive portfolios tend to be held for longer periods. Because adjustment is not as frequent as for active portfolios, assets need to be chosen to meet the strategic aims of the portfolio rather than short-term goals.

2.2 Passive portfolio management

Passive (bond) portfolios are generally held for the medium to long term.[1] The return on the portfolio depends in general on the yield of the assets in it. Adjustments are made generally when cash inflows or outflows occur, or for strategic reasons (such as a change in economic view). Bonds may be held to maturity. The portfolio manager will aim to get the cashflows in the portfolio to match required outflows, or ensure assets are liquid enough to provide them. Day-to-day market changes will affect the instantaneous value of the portfolio. This may not disrupt the overall investment strategy if obligations are met. They will determine the yield at which assets are purchased, and valued. Cash inflows will be invested to fit the overall time horizon and strategy that is adopted for the portfolio.

The main advantage of passive portfolios are that they do not require a lot of short-term adjustment. This cuts down on both transaction costs, and on the resources required in their management. Provided the assets held in a passive portfolio meet both cashflow and investment return obligations, the risks inherent in these are low. Management is not necessarily (or overly)

1. There is no firm delineation between short-, medium- or long-term portfolios. Long-term portfolios can be constructed over years or decades. Medium-term can range from weeks or months to several years, while short-term portfolios can change daily or even intraday.

concerned with day-to-day fluctuations due to market volatility. More important is the overall economic and strategic outlook. The passive portfolio manager may not need to hold liquid assets (however, under disaster situations this could be a problem). This means that access to high-yielding bonds will increase, as assets need not be benchmarks—or have liquidity constraints. Credit exposure becomes important as bonds are held for longer periods.

Passive portfolios require adjustment periodically. This will be due to overall strategic considerations rather than short-term trading opportunities (for example, there may be asset allocation changes, the overall economic view of the manager may be modified, or timing of inflows and outflows needs to be altered). The quality of assets held must reflect the likely holding period.

It is useful for the passive portfolio manager to have, or be aware of, disaster recovery plans. These can be utilised where unforeseen events occur. Such scenarios include market catastrophes—yields plummet or soar—or where a sizeable portion of the portfolio needs to be liquidated. They also include contingencies for issuer defaults or missing interest payments. These can be alleviated by the using of short-term hedges, having the capacity to borrow (money or assets), substituting assets, or by keeping a portion of the portfolio in liquid and tradable assets including cash.

2.3 Active portfolio management

An actively managed portfolio is usually constructed to take advantage of short-term market moves. These are more often relative than absolute. They include changes in the shape of the yield curve or relative movement between different yield curves. As such, actively traded portfolios are often constructed insensitive to overall uniform shifts. They exploit the changes in relativity between assets. Hedging becomes very important. The portfolio needs to be insulated against all changes except those which the manager chooses to exploit. An example of this is where the manager perceives the yield differential between two instruments to be excessively large. To exploit this, the higher-yielding asset will be bought, the lower-yielding sold (or shorted). If this is done so that the portfolio is insensitive to absolute movement of the two bonds, then the only exposure is this relativity.

Active portfolio returns can be measured against a benchmark, or they may just aim to maximise outright return (on capital). Benchmarks are usually market indices, or specific bonds. The aim of the manager is to use short-term trading opportunities to beat the benchmarks or to maximise return. Performance will include transaction costs, which can be significant. Active management of a portfolio must fit within predefined risk limits. These may govern the outright exposure or the exposure to certain assets and parts of the yield curve.

Actively managed portfolios may be used either to generate maximum trading profit, or they may also have some strategic component. In the former case, the manager need not focus on the size and timing of cash inflows and outflows. Profitability is more linked to changes in capital value. Often these portfolios are geared,[2] so funding costs need to be taken into account. Where there is some strategic component, the duration of assets will also need to be managed.

3. BENCHMARKING

The process of benchmarking enables the performance of a portfolio to be measured. It also gives the manager a target for which performance is to be achieved or exceeded. There is no unique benchmark. Generally, a performance benchmark will be the risk/return of an entire market sector, that of a single asset, or an outright return on capital target.

3.1 Single instrument benchmarks

A single instrument benchmark is a bond portfolio where the target return is, for example, that of a ten-year Government bond. The manager must maintain modified duration within (say) one year of the ten-year bond modified duration. Within this constraint, the manager can exactly meet the benchmark if the portfolio consists of just the benchmark ten-year bond, with interest cashflows reinvested in this bond. The major risk then is reinvestment risk. Management of this portfolio would be very passive. The only adjustments would occur when there is a coupon to be reinvested. An active approach would be to hold a series of bonds of maturities around ten years, but not necessarily the ten-year benchmark itself. The manager must choose bonds so that the combination return including costs matches or exceeds the benchmark, whilst maintaining modified duration within the required range around the benchmark. There is greater risk associated with this strategy, however returns can be significantly enhanced if the manager performs well.

Other common benchmarks of this type include cash, or short-dated bills. The actual instruments in the portfolio may or may not be the same as the benchmark itself.

3.2 Market sector or index benchmark

Using indices, the aim of the manager is to construct a portfolio which will replicate or exceed returns of a complete market or market sector. Again, the passive manager can hold (or replicate) the index. The active manager has choice of instruments—usually, but not always—in the sector or index. In the case where the manager is measured against a specific index, but can hold instruments not contained in the index, considerable excess risk may be introduced by the wider choice of instruments available. The return target of the index may become easily attainable if the portfolio can contain other instruments. A common example of this is where a portfolio is benchmarked

2. A geared portfolio is one in which cash is borrowed to buy assets. This means that market exposure can be greater than the capital value of the portfolio.

against a Government bond index. The portfolio is not restricted to holding only Government bonds. Under normal conditions, the portfolio manager can choose to hold higher-yielding corporate bonds. This means that usually returns will exceed the index almost by default. What is not considered however is the additional credit risk associated in holding the more risky corporate bonds.

Sometimes a portfolio is broken down into sub-indices. This is common for very large portfolios, where more than one manager is employed. Consider a very large pension fund whose benchmark is the Government bond index. The portfolio is split into sub-portfolios of various maturities. These may include a cash or bank bill component for durations of 180 days or less, a zero to three year, three- to seven-year and greater than seven-year components. Each is managed against a relevant sub-index. Each contains instruments similar to the sub-index. There may be some overlap. The entire portfolio can be actively managed. The overall strategy is of deciding what proportion of the total portfolio value to allocate to each sub-group. The decision is whether to allocate the portfolio exactly as maturities are apportioned in the index, or to overweight certain maturities. If the overall view is that long-term assets will perform better than short-term ones, the decision might be made to hold less cash and bill type assets in favour of excess weighting in the over seven-year area. Once this strategic allocation is made, each sub-portfolio is managed separately. This gives the managers of these the choice to tilt the portfolio toward areas where overperformance is anticipated. This top down approach provides a number of areas which can be managed. The top level allocation may fit the index, while the sub-portfolios may be tilted, or vice versa.

3.3 Example: top down portfolio management

From the above discussion, consider a large portfolio which is managed against an all maturities index. The table below shows the allocation of the complete portfolio under differing strategic viewpoints.

Maturity band	Average modified duration (y)	Index Weighting	Neutral	Rates increasing	Rates decreasing
Cash—180 days	0.25	16.0%	16.0%	20.0%	14.0%
180 days-3 years	1.6	31.5%	31.5%	34.0%	29.0%
3-7 years	4.4	24.5%	24.5%	22.0%	27.0%
Over 7 years	8.9	28.0%	28.0%	24.0%	30.0%
Total		100.0%	100.0%	100.0%	100.0%
Modified duration		4.11	4.11	3.70	4.36

The table shows the components of the index. A neutral portfolio would exactly mirror this. If the strategic view were that rates were going to increase, then, in general, long-term bonds would perform worse than short-term bonds. The portfolio under this scenario is tilted towards the short end. The modified duration has been shortened from 4.11 years to 3.7 years. The converse is true for the view of rates decreasing. The modified duration is lengthened by 0.25 to 4.36 years.

The above is the overall top level allocation of the portfolio. If the sub-portfolios are separately managed, then there will be tilting within them as well. This can increase or decrease the tilt generated by the top level allocation. If from the above example, the overall allocation is performed with the view that rates will increase, then the sub-portfolios are given a certain proportion of the total portfolio value. The sub-managers may decide to further tilt. If limits are set on the amount each sub-portfolio can be altered from the index, then this effect will be secondary. It may be that the overall asset allocation can move the modified duration plus or minus half a year from the index. The allocation allowed in any sub-portfolio may only be allowed to alter the entire portfolio (including all other sub-portfolios) by plus or minus 0.1 years.

There may be other levels in the asset allocation hierarchy that have not been discussed here. These include global and sector allocation. Global allocation determines what the geographic composition of a portfolio should be. What exposures to what countries or regions are required. Also within a sector, there is the choice of assets. This will include classes such as equities, property, commodities or bonds. It is only the latter that is considered in detail here.

3.4 Return on capital benchmark

As distinct from benchmarks which are an instrument or index, a portfolio may be managed to provide a certain return on capital. Such is the case for many hedge funds and other portfolios with aggressive investment profiles. An outright target return may be set, or the portfolio will aim to achieve a certain margin over cash (or another asset). In this case, the risk profile is usually much higher than for an index benchmarked portfolio. The portfolio will be aggressively managed. There may be little restriction on the instruments the portfolio can hold, and the types of trading activity permitted.

4. CASE STUDY: HEDGING A BOND PORTFOLIO

The following discussion focuses on the hedging of a bond portfolio. It is assumed that the asset allocation and asset choices have been made. Here we show how an asset manager can insulate the portfolio to changes in the market rates (the yield curve). We will use a relatively simple portfolio for this study. These principles can easily be extended to more complex portfolios.

4.1 The portfolio and the yield curve

For simplicity, the portfolio in this study will hold just three bonds. Their characteristics are shown below.

	Bond 1	Bond 2	Bond 3
Spot date	1/6/97	1/6/97	1/6/97
Maturity	15/11/99	15/7/04	1/8/08
Coupon	10.00%	6.50%	8.25%
Coup freq	2	2	2
Yield	7.25%	7.93%	8.12%
Price (per $100)	106.536	94.780	103.659
PVBP (per $m)	114.867	255.392	365.106
DMod	1.083	2.766	3.618
Convexity	0.016	0.087	0.176

The composition and overall characteristic of the portfolio is shown.

	Face val ($m)	Value ($)	PVBP	DMod	Convexity
Bond 1	2	2,130,720	229.734	1.083	0.032
Bond 2	2	1,895,600	510.784	2.766	0.174
Bond 3	6	6,219,540	2,190.636	3.618	1.056
Total	—	10,245,860	2,931.154	2.933	1.262

The portfolio has a total value of just over $10 million. Its modified duration is 2.933 years and the PVBP or sensitivity to a one basis point change in yield is $2931.154.

As hedge instruments, we have available two zero coupon bonds maturing on 15/12/00 and 15/12/06 respectively. The characteristics of these are shown below.[3]

	Hedge 1	Hedge 2
Spot Date	1/6/97	1/6/97
Maturity	1/12/00	1/12/06
Coupon	0.00%	0.00%
Yield	7.50%	8.05%
Price (per $100)	77.283	47.248
PVBP (per $m)	130.369	215.796
DMod	1.687	4.567
Convexity	0.025	0.104

To hedge the portfolio, various combinations of the hedge instruments can be sold (or shorted). This will neutralise some of the sensitivities of the portfolio. There are many choices of combinations for the hedge instruments. It is assumed that the proceeds of the hedge instruments when sold are invested in cash or other readily liquid assets. Funding costs are not considered here. We shall assume that funding costs are negligible—and so do not influence the choice of hedge.

The extremes in the choice of hedge will be all of one or the other zero coupon bonds. Any other combination can be chosen. Generally the

3. The zero coupon bonds are priced as if they had a frequency of two coupons per year. This makes their yield and pricing consistent with that for the bonds in the investment portfolio.

parameter that affects the value of the portfolio most is changes in the yield curve. If the portfolio plus hedge is PVBP neutral, then this will be insulated under small parallel shifts. We consider the hedge performance under a number of scenarios.

4.2 Parallel shifts

Using each hedge instrument separately, the following hedge portfolios can be constructed.

Zero Coupon Hedge Characteristics

	Face val ($m)	Value ($)	PVBP	DMod	Convexity
Portfolio		10,245,860	2,931.154	2.933	1.262
Hedge 1	−22.4835	(17,375,939)	(2,931.154)	1.687	0.562
Hedge 2	−13.583	(6,417,689)	(2,931.154)	4.567	1.413

This means that either $22,483,500 face value of the zero coupon maturing on 1/12/00, or $13,583,000 face value of the zero coupon bond maturing on the 1/12/06, can be used as a hedge. In both cases, the hedge PVBP exactly cancels the PVBP of the portfolio. We now examine what happens to the portfolio and the hedges under various parallel yield curve shifts.

Performance of Hedge 1 Under Parallel Shifts

	Portfolio change ($)	Hedge 1 change ($)	Difference ($)	Diff/bp
−100bp	612,173.41	(597,640.41)	14,533.00	145.33
−10bp	58,863.24	(58,730.69)	132.55	13.25
−1bp	5,863.57	(5,862.88)	0.68	0.68
zero	—	—	—	—
+1bp	(5,858.53)	5,860.62	2.09	2.09
+10bp	(58,359.43)	58,504.69	145.27	14.53
+100bp	(561,744.81)	575,036.94	13,292.14	132.92

Performance of Hedge 2 Under Parallel Shifts

	Portfolio change ($)	Hedge 2 change ($)	Difference ($)	Diff/bp
−100bp	612,173.41	(615,234.15)	(3,060.74)	(30.61)
−10bp	58,863.24	(58,891.66)	(28.43)	(2.84)
−1bp	5,863.57	(5,863.72)	(0.15)	(0.15)
zero	—	—	—	—
+1bp	(5,858.53)	5,858.08	(0.45)	(0.45)
+10bp	(58,359.43)	58,328.25	(31.18)	(3.12)
+100bp	(561,744.81)	558,842.74	(2,902.06)	(29.02)

The tables above show the difference in value of the investment portfolio and the hedge under parallel shifts (where all yields change by the same amount). For small shifts (of plus or minus one basis point), the difference is small. The hedge performs well by countering the change in value of the portfolio. Where the shifts are more extreme (plus/minus 100 basis points), there is significant difference between the change in value of the portfolio and the hedge. Even when expressed as a per basis point change, it is significant.

4.3 Non-parallel shifts

Although the hedge performance is less than perfect when there are large parallel shifts, the hedge still offers a large amount of protection. The worst case above, the unhedged portfolio changes in value by $612,173, while the worst hedge decreases this to less than $15,000.

What happens when the yield curve shifts in a non-parallel manner? To illustrate this, we consider steepening and flattening of the curve.

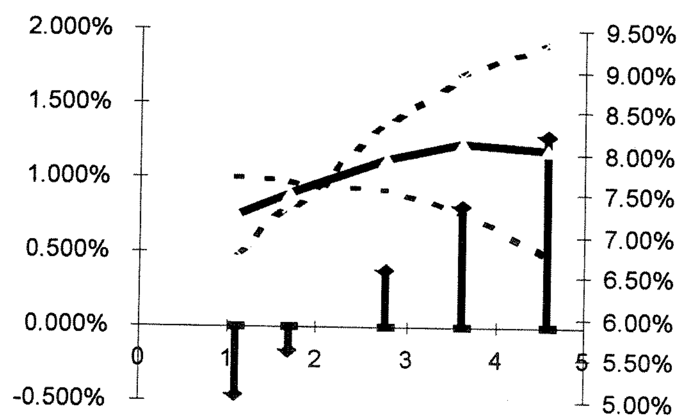
4.4 Digression: non-parallel yield curve movement

Over short time periods, the yield curve changes. A hedged bond portfolio tries to insulate the effects of these changes from the underlying portfolio. It will never be possible to completely remove sensitivity to change. In practice, hedge strategies are implemented to remove the gross sensitivities. Analysis has shown that most short-period movement of the yield curve is parallel. PVBP matching removes most of this sensitivity. The next most important effect is due slope changes of the yield curve. It is hard to specify exactly how the curve steepens or flattens. Are these changes in the yield curve or the zero curve? Is there a pivot point, or does the curve shift as well? To explore the effect of slope, we will use the modified duration curve as our yield curve. We apply linear slope changes to the curve using the two-year point as a pivot. To gauge effects where the pivot is elsewhere, a parallel shift can be combined with the slope change.

Slope changes (measured in basis points per year) can be superimposed on the modified duration curve. Sample changes are shown on the curve below.

Exhibit 13.1

The Modified Duration Curve with Steepening and Flattening



The original portfolio and the hedge portfolios are revalued under these scenarios.

Performance of Hedge 1 Under Curve Slope Changes

	Portfolio change ($)	Hedge 1 change ($)	Difference ($)	Diff/(bp/y)
+50bp/y	(1,282,135.25)	(371,522.82)	(1,653,658.06)	(33,073.16)
+10bp/y	(358,305.51)	(92,040.48)	(450,345.99)	(45,034.60)
zero	—	—	—	—
−10bp/y	387,200.20	91,486.64	478,686.85	47,868.68
−50bp/y	1,749,027.50	362,660.88	2,111,688.38	42,233.77

Performance of Hedge 2 Under Curve Slope Changes

	Portfolio change ($)	Hedge 2 change ($)	Difference ($)	Diff/(bp/y)
+50bp/y	(1,282,135.25)	2,379,337.48	1,097,202.23	21,944.04
+10bp/y	(358,305.51)	707,856.75	349,551.24	34,955.12
zero	—	—	—	—
−10bp/y	387,200.20	(800,830.20)	(413,630.00)	(41,363.00)
−50bp/y	1,749,027.50	(3,899,959.71)	(2,150,932.22)	(43,018.64)

These changes are now very large. The hedge is not very effective against non-parallel curve changes.

4.5 Duration matching

Under parallel shifts, a single hedge instrument performs well when the move is small. This is due to the significant difference in the structure of the portfolio to that of the hedge. A significant improvement can be attained if a mix of the hedge instruments is used. If a combination is used such that both the PVBP and the modified duration of the hedge and the original portfolio are the same, the hedge is as follows.

Combination Hedge

	Face val ($m)	Value ($)	PVBP	DMod	Convexity
Portfolio		10,245,860	2,931.154	2.933	1.262
Hedge 1	−7.335	(5,668,843)	(956.279)	1.687	(0.183)
Hedge 2	−9.152	(4,323,939)	(1,974.875)	4.567	(0.952)
Hedge Total	—	(9,992,781.936)	(2,931.154)	2.933	(1.135)

Under the parallel shift scenarios, the hedge performance is marginally better than the single instrument hedges. There is still a mismatch when the shifts are large.

Performance of Combination Hedge Under Parallel Shifts

	Portfolio change ($)	DMod Hedge change ($)	Difference ($)	Diff/bp
−100bp	612,173.41	(609,494.25)	2,679.16	26.79
−10bp	58,863.24	(58,839.15)	24.09	2.41
−1bp	5,863.57	(5,863.45)	0.12	0.12
zero	—	—	—	—
+1bp	(5,858.53)	5,858.91	0.38	0.38
+10bp	(58,359.43)	58,385.81	26.39	2.64
+100bp	(561,744.81)	564,126.05	2,381.24	23.81

When the non-parallel shifts of the previous section are applied, the duration-matched hedge performs significantly better than the single instrument hedges.

Performance of Combination Hedge Under Curve Slope Changes

	Portfolio change ($)	DMod Hedge change ($)	Difference ($)	Diff/(bp/y)
+50bp/y	(1,282,135.25)	1,481,878.37	199,743.12	3,994.86
+10bp/y	(358,305.51)	446,892.92	88,587.41	8,858.74
zero	—	—	—	—
−10bp/y	387,200.20	(509,714.77)	(122,514.57)	(12,251.46)
−50bp/y	1,749,027.50	(2,509,293.78)	(760,266.29)	(15,205.33)

Even with this duration-matched hedge, there is still significant deviation between the investment portfolio and the hedge.

4.6 Further improving the hedge portfolio

The portfolio in the above case study was hedged using only two zero coupon instruments. These differ significantly from the underlying portfolio. The timing and size of cashflows is different. They are sensitive to different parts of the yield curve. Despite this, they are effective under parallel shifts, and partially effective with slope changes. There are a number of ways in which hedging can be improved.

4.7 Scenario analysis and optimisation

If more hedge instruments are available, an optimal hedge can be constructed using scenario analysis. This is usually a numerical optimisation procedure. The inputs are the characteristics of the underlying portfolio and the hedge instruments. A number of yield curve scenarios are defined. These include the most likely changes to the curve, plus a few disaster scenarios (large moves or shape changes). The scenarios can be probability weighted. The optimisation procedure finds the portfolio that minimises the mismatch between the hedge and the underlying portfolio. This technique, although numerically complex, can yield very effective results.

4.8 Dynamic hedging

The portfolios in this case study were set up with *static hedges*. Once set up, they were not altered—even where the yield curve changes were large. In reality, the hedge can be adjusted periodically. This is called *dynamic hedging*. A dynamic hedge is rebalanced whenever the value of the portfolio changes. Changes can be due to natural time effects (such as interest payments and maturing bonds), or due to yield movement. Various criteria can be set as to when a rebalance should be performed. These can be a dollar mismatch, or a yield curve change.

With frequent rebalancing, hedging can be very effective—even where the hedge instruments differ markedly from the underlying portfolio. A trade off must be made between hedge cost (both of the hedge portfolio and the resources required to manage it) and effectiveness.

4.9 Use of derivatives

Another method for hedging is by the use of derivatives. Futures contracts are the simplest form of derivative instrument commonly used for hedging. Their use is so widespread that they are often treated as vanilla instruments. More complex derivatives such as options or forward contracts can be very effective in removing risk. They can also allow a portfolio manager to capture upside which would normally be lost with vanilla hedges (for example, an option can protect against interest rate rises whilst allowing the portfolio to profit from rate falls). The main problem with using derivatives is their cost. Options usually have a large premium associated with them. This needs to be considered within the overall portfolio management brief.

Chapter 14

Portfolio Insurance

by Steuart Roe

1. INTRODUCTION

The aim of this chapter is to provide an introduction to the main concepts of portfolio insurance. Portfolio insurance is a generic term used to describe investment strategies that involve the capital protection of an investor's funds. For example, an investor may not be prepared to accept a negative return over a one-year period.

The simplest way to insure an investment is to buy a put option over the investment, struck at the current value of the investment. The put option protects the investor from capital depreciation on the investment by paying the investor the depreciation, if any, at the expiry of the option. *Exhibit 14.1* shows the value of an insured investment at the expiry of the option and the value of the same investment without protection as the value of the underlying investment varies. The difference between the insured investment and the uninsured investment, if the portfolio value rises, is the cost of the option or insurance.

When executing a portfolio insurance program over a portfolio of investments there are many things a fund manager must consider. For instance, how much will it cost? Should the option be purchased or replicated? If an option is purchased, would it be exchange-traded or over-the-counter? How will I pay? How credit worthy is the option seller? What documentation is required? What if I change the composition of my underlying assets? How will I monitor my exposures? How will cash flows be insured? Et cetera. The answers to these questions are rarely black and white.

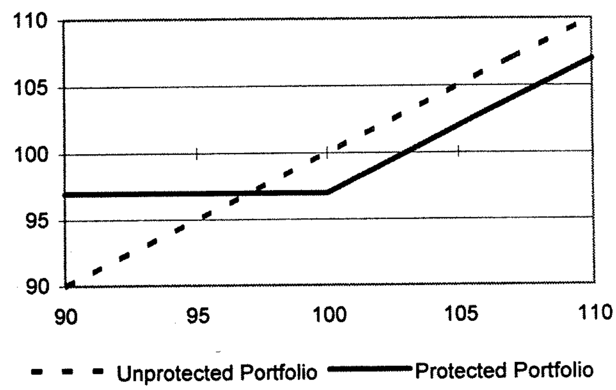
The chapter examines:

- option-based portfolio insurance;
- cost of protection;
- synthetic option replication;
- constant proportion portfolio insurance; and
- risks associated with portfolio insurance.

Exhibit 14.1

Pay-off from an Uninsured Portfolio vs an Insured Portfolio

$$\sqrt{\sigma^2\left(1+\frac{2k\sqrt{n}}{\sigma\sqrt{T-t}}\right)}$$



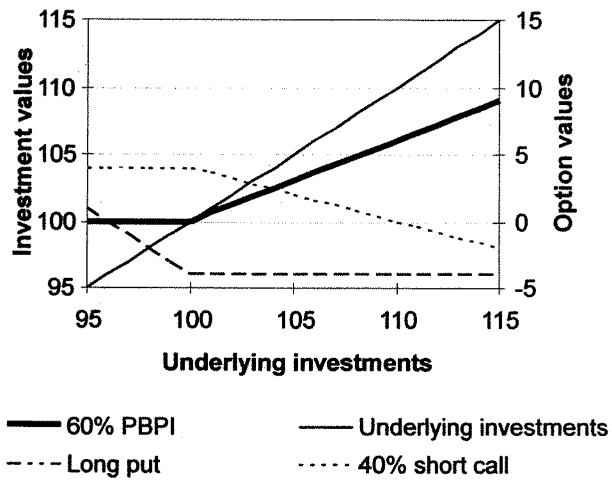
2. OPTION-BASED PORTFOLIO INSURANCE

When choosing an insurance program the first decision is to decide what and when the payoff is required. A minimum return objective needs to be set, known as the floor. The second question is to decide how the insurance or put option is to be paid. One way to pay for the put option is to sell call options.

Let us suppose an investor wanted to ensure that in one year's time his investments were at least as valuable then as they are today, that is, a floor of 0%. If put and call options struck at the current value of investor's assets cost 4% and 10% respectively then the investor could pay for the put option by selling 40% of the call option. By selling 40% of the call option the investor has limited the capture of any increases in the value of the investments to 60%. *Exhibit 14.2* shows how the investor's portfolio of investments plus a put option less 40% of a call option combine to give the investor a zero floor and a participation rate of 60%. Paying for insurance in this manner is generally referred to as Participation-based Portfolio Insurance (PBPI).

Exhibit 14.2

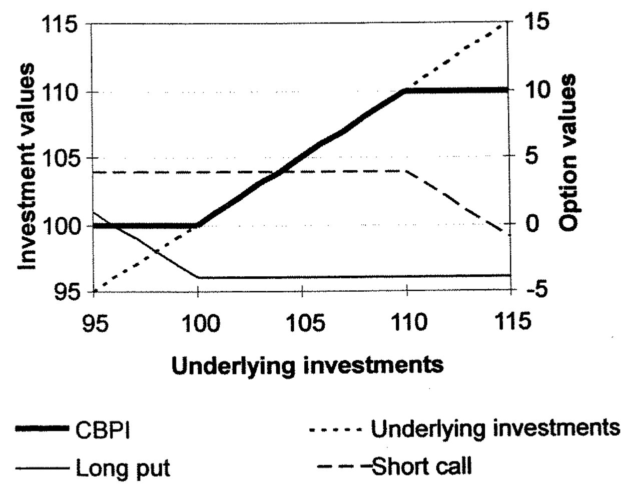
Participation-based Portfolio Insurance



An alternative to selling call options struck at the same level as the put option is to sell a call option struck such that its value is equivalent to the value of the put option. This method of paying for the protection *caps* the total return the investor can achieve to the strike of the call option. For instance, if, using the previous example, a call option struck at 110% of the investor's assets was worth 4% then the investor could sell this call option to pay for the put option. *Exhibit 14.3* shows how the investor's portfolio of investments plus a put option less a call option struck at 110% combine to give the investor a zero floor and 100% of any upside with the performance capped at 10%.

Exhibit 14.3

Capped-based Portfolio Insurance



There are in fact an infinite number of ways the protection can be paid for. Below, I have formally defined three different strategies that provide a fixed percentage of any upside above a floor for some time in the future. For conceptual simplicity, they refer to combinations of cash and call options to construct their desired pay outs. Other combinations using put and/or call options could be created using put-call parity to provide the same pay off.

Each of these strategies involve an investment in a zero coupon bond, maturing on the same day as the protection, with a face value equivalent to the floor. The difference between the face value of the zero coupon bond and the original investment is the amount of money the investor can lose and still meet the capital protection objective. This difference is known as the *cushion*. The cushion is then invested in call options. How many call options (n) and where the call options are struck (K) is dependent upon the desired pay off.

The first of these strategies is known as Option-based Portfolio Insurance (OBPI) and is defined by the following equations:

$$nC_{BS}(P_0,K,T - t,\sigma,i) = P_0 - F_T(1 + i)^{-(T-t)}$$

$$nK = F_T \qquad \text{.... (OBPI)}$$

Where

C_{BS} = standard Black-Scholes value for a call option;

P_0 = portfolio value at time of the start of the insurance program;

F_T = floor value of portfolio at maturity of the insurance program;

σ = volatility of the underlying asset; and

i = annualised risk free rate.

The first equation ensures the entire cushion is invested in options, the second that the total exercise cost equals the floor—thus preventing gearing.

If the desired pay off is to participate in some percentage (n) in the value of the underlying portfolio from the floor level then the strike should equal floor. I refer to this strategy as Participation-based Portfolio Insurance (PBPI).

$$K = F_T \qquad \text{.... (PBPI)}$$

It is sometimes necessary to place a further constraint on this strategy such that if the floor is less than the existing value of the underlying assets then the pay off is a percentage of the return. This constraint ensures that a zero return on the underlying assets results in a zero return on the insured portfolio.

$$F + n(P_0 - K) = P_0 \qquad F_T < P_0$$

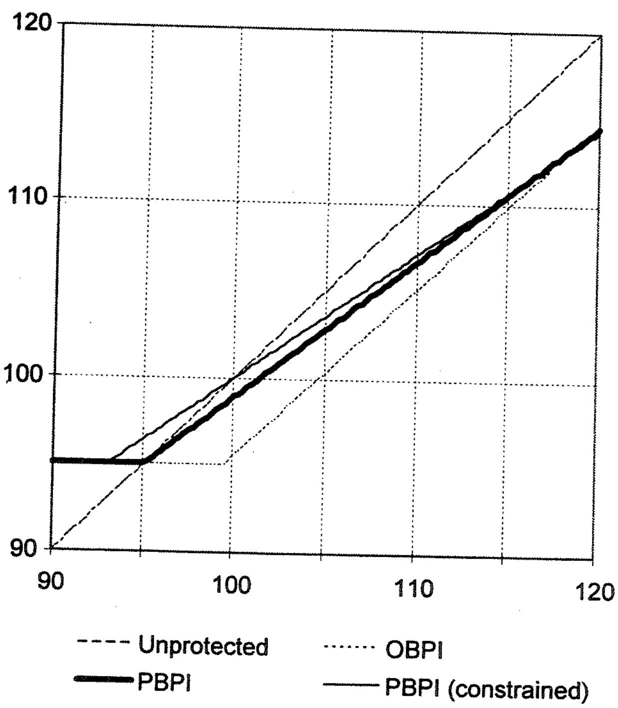
.... (PBPI constrained)

$$K = F_T \text{ otherwise}$$

Exhibit 14.4 plots the pay off structure for each of these strategies. It assumes P_0 is 100, F_T is 95, volatility is 20%, an annualised risk free rate of 7% and a one year term.

Exhibit 14.4

An Uninsured Portfolio vs Option-Based Protection Strategies



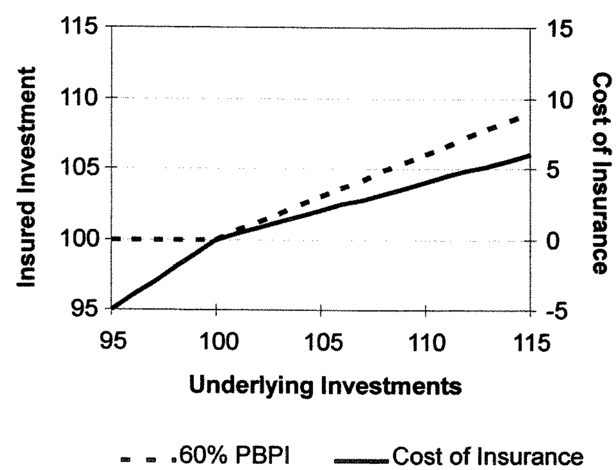
3. COST OF INSURANCE

Once the floor and the method of paying for the insurance premium have been determined, the investor needs to be satisfied as to both the short- and long-term cost of insurance. The cost of insurance is the value of an uninsured portfolio less the same portfolio insured. Likewise, the expected cost of insurance is the expected return on an uninsured portfolio less the same portfolio insured.

Note the distinction between the insurance (or put) premium, the expected cost of insurance and the actual cost of insurance. For example, if the cost of a put option on an investment was 4%, then the expected cost of insurance would be less than 4%, as there is some possibility the insurance will be called. *Exhibit 14.5* shows how the actual cost of insurance varies for a PBPI strategy with 60% participation. If the underlying asset rises then the insurance in that year is an expense to the investor. Conversely, if the underlying assets fall, then the insurance pays off, so the actual cost of protection is negative, that is, there is a claim against the insurance. For instance, a 10% rise and fall result in actual costs of insurance of 4% and minus 10% respectively.

Exhibit 14.5

Actual Cost of Insurance



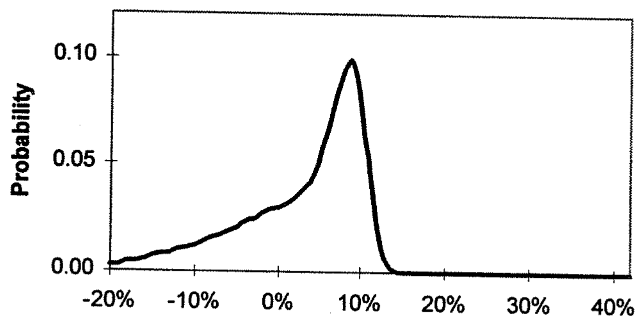
An approximation for determining the expected return on an insured portfolio is to multiply the delta equivalent of uninsured holdings by their risk premium and add the risk free rate.

$$\begin{aligned} E(\text{cost of insurance}) &= E(\text{uninsured return}) - E(\text{insured return}) \\ &= r_f + r_p - (r_f + \Delta r_p) \\ &= r_p (1 - \Delta) \end{aligned}$$

Of course, in any given period the actual cost of protection can be varied. *Exhibit 14.6* shows the distribution of the cost of protection for a CPPI (referred to later in the chapter) program using the same assumptions in *Exhibit 14.4*. A negative cost of protection means the protection paid off.

Exhibit 14.6

Distribution of the Cost of Insurance



If the variability or the expected cost of insurance are unacceptable to the investor then the floor and/or the method for paying for the insurance premium will need to be modified.

4. SYNTHETIC REPLICATION

After determining the desired pay off the next question to answer is: how to implement the strategy? Probably the easiest approach would be to buy and/or sell the necessary options. Options on a wide range of securities can be traded on the various exchanges around the world. Options are available on individual company shares, stock market indices such as the S&P 500, fixed interest securities, et cetera. Furthermore, there are a large number of investment banks that will buy and sell options on an over-the-counter basis on just about anything.

However, occasionally there may not be any sellers or buyers of the desired options or, more likely, the desired options are either too expensive or too cheap. An initial attempt at deciding an options relative value is to compare the volatility implied by the options price to the expected future volatility of the returns on the underlying asset. When this is significantly different an alternative to trading the options is to replicate the options.

The standard option replication technique is called delta hedging. Delta hedging requires an amount of the asset, which the option is on, to be purchased or sold (known as the hedge) such that a small movement in the price of the asset results in the same value change in the hedge and the option. Theoretical option pricing models, such as the Black-Scholes model, allow the hedge or delta amount to be estimated.

4.1 Example 1: delta hedging

Calculate the value of a one year European at-the-money call option over an asset currently trading at $100 given that the one year annual interest rate is 7% and the expected volatility of the underlying asset is 15%.

$$C_{BS}(100,100,1,15\%,7\%) = 9.6$$

Given these assumptions and the underlying assets value in the table below, calculate the pay out from replicating the option by re-hedging the delta weekly.

The Replication column represents the value of a portfolio that has sold the call option. At time zero $9.6 is received for selling the call option, $60.5 (70.1 − 9.6) is borrowed and delta times the value of the underlying asset $70.1 (0.701 * 100) is bought. At the end of the first week the underlying asset has fallen to $98 so that the value of the portfolio at the end of the week is worth the value at the start of the week plus the appreciation in the asset value less the cost of any borrowings. In this case the value of the portfolio at the end of the first week is:

9.6 + 70.1 * (98/100 − 1) − (70.1 − 9.6) * ((1.07) ∧ (1/52) − 1) = 8.2

Week	Asset	Delta	Equity	Replication
0	100	0.70	70.1	9.6
1	98	0.65	63.7	8.2
2	96.7	0.61	59.3	7.2
3	96.2	0.60	57.4	6.9
4	98.2	0.65	63.6	8.0
5	99.3	0.67	66.9	8.6
6	102.3	0.74	76.1	10.6
7	106.2	0.82	87.2	13.4
8	107.2	0.84	89.9	14.1
.				
.				
.				
50	115.1	1.00	115.1	15.6
51	116.2	1.00	116.2	16.6
52	118.5			18.7

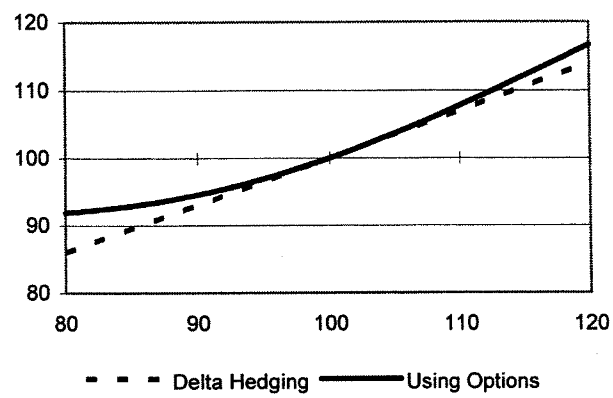
Because the assets value at the end of the year is $118.5, the call option should be worth $18.5. Replicating the option produced a pay-out of $18.7, very similar to the pay-out from the option. This is essentially because the actual volatility of the underlying asset proved to be very similar to the volatility predicted.

The problem with delta hedging is it is dependent upon the same assumptions as the Black-Scholes formula. Most critically, it is dependent upon a continuous and frictionless market. That is, the market does not gap up or down, there is no impact on market prices due to trading and there are no transaction costs. In the real world these assumptions are rarely valid.

The possibility of discontinuous markets (that is, market gaps) increases the value of an option. This is because an option seller risks the possibility that the market gaps. In return for this risk the seller will require a premium. Exhibit 14.7 shows the value of a portfolio attempting to replicate the pay-off of a call option using delta hedging and the pay-off of a portfolio of cash and the same option if the underlying asset were to jump to the level in the x-axis. Note that if the asset were to fall 20% then the value of the delta hedging portfolio would fall to 86 while the value of the portfolio containing the option would only fall to 91. This risk may not be acceptable to an investor considering replicating an option.

Exhibit 14.7

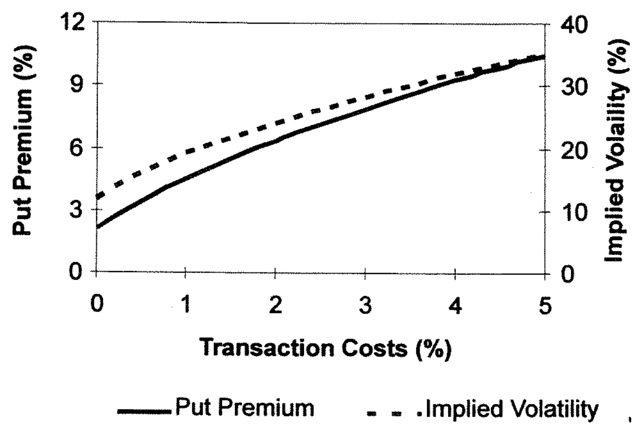
Delta Hedging vs Options



Where transaction costs exist, they also increase the value of options. The impact on the cost of an option and the option's implied volatility for different transaction costs is approximated in *Exhibit 14.8*. The chart clearly illustrates the sensitivity of an option's value to transaction costs.

Exhibit 14.8

Implied Volatility vs Transaction Costs



The chart uses the volatility approximation technique developed by HE Leland for incorporating transaction costs

$$\sqrt{\sigma^2\left(1+\frac{2k\sqrt{n}}{\sigma\sqrt{T-t}}\right)}$$

Where k is transaction costs, n is the rebalancing frequency. The chart assumes σ = 12%, n = 83 and T-t = 1.

Another important factor to consider when deciding whether to replicate or not is the management and infrastructure required. Replication requires constant monitoring of positions and exposures, sophisticated computer software and hardware and qualified implementation staff. Each of these cost money. The purchase of an option can be viewed as a mechanism to outsource the management and infrastructure of replicating the option together with guaranteed performance.

Hence, the decision to replicate or not should be based upon a comparison between the value of the option being offered, if available, and the theoretical option value based on future expected volatility plus a margin for transaction costs plus a margin for gap risk plus a margin for management and infrastructure.

4.1.1 Futures

One way to significantly improve the efficiency of a hedging program is to use futures contracts as the hedging instrument. This is because futures normally have far lower transaction costs than their physical equivalents. The amount of futures contracts to be traded to hedge a given amount of underlying is given by:

futures contracts = hedge/futures equivalent

= hedge/(index * contract multiplier * $e^{(i-d)(T-t)}$)

Where

i is the continuous risk free interest rate;

d is the continuous dividend rate; and

T-t is the term of the futures contract.

4.2 Example 2

In Example 1, the delta on the option to be replicated was 0.701 at the start of the period. Using the same assumptions as example 1 calculate the amount of futures contracts required to hedge 1000 of these options if the futures contract expires in three months and the contract multiplier is 25.

*1000 * 0.701 * 100/(100 * 25 * (1.07) ∧ (1/4)) = 27.6*

The problem with futures contracts is that they may not be available on the assets to be hedged. One way to compensate for this problem is to use futures contracts on a similar asset or assets and *beta* adjust the amount to be traded. The equation for the amount of futures contracts to be traded to hedge a given amount of underlying becomes:

beta * hedge/futures equivalent

In addition, if futures are used to hedge a portfolio that does not exactly match the futures contract then, even with beta adjusting the hedge, tracking error may arise. In a dynamic hedging approach to portfolio insurance the tracking error can be built into the management. For instance, a portfolio insurance program could be run using index futures as the hedging instrument for an actively managed portfolio of equities. If the equities outperform the index, after beta adjustments, then the strike of the option relative to the index can be lowered. Hence the outperformance serves to reduce the cost of protection. Conversely, and even more importantly, if underperformance occurs then the strike must be raised if the capital protection objective is to be met.

4.2.1 The 1980s experience

During the early to mid-1980s portfolio insurance experienced phenomenal growth in popularity. Funds under management in the US grew from nothing to an estimated $90 billion. When the stock market crashed in October 87 some funds employing portfolio insurance were unable to meet their desired level of protection. In fact, the Brady Commission report into the crash of 1987 found that portfolio insurance was partly responsible for the crash due to the large volume of selling required by the portfolio insurers because of their hedging requirements. As a consequence of this combination of factors, portfolio insurance fell from grace and became a "dirty" word in the investment community. Proponents of portfolio insurance have disputed these findings.

Nonetheless, portfolio insurance is alive and well in some form or other today. This is due to the risk-averse nature of people. In fact, all rationale investors would prefer to have all the upside with none of the downside of investing. It just boils down to the cost of protection versus their risk tolerance.

5. CONSTANT PROPORTION PORTFOLIO INSURANCE

In 1986, Fisher Black published an article proposing Constant Proportion Portfolio Insurance (CPPI). The aim of CPPI is to provide a portfolio insurance technique that protects investors from gaps in markets. CPPI uses a simplistic methodology that does not use option pricing theory and is intuitively appealing. It works as follows:

1. *Establish the floor price*. The floor price is the level that the portfolio's value could fall to and still meet the minimum return objective, if all the assets of the fund were invested in a zero coupon bond that matured on the same day as the insurance program.
2. *Calculate the cushion*. The cushion is the difference between the current value of the portfolio and the floor. The cushion represents how much money the fund can lose at the instant of calculation and still meet the minimum return objective.
3. *Determine exposure to risky assets*. The risky assets are the assets the portfolio is investing whose return the portfolio is aiming to capture. The exposure to the risky assets is the maximum amount that can be invested in the risky assets and still meet the minimum return objective of the portfolio. It is determined by dividing the cushion by the percentage fall (or crash factor) the investors are attempting to protect themselves against from the risky assets. Hence, if the risky assets were to instantaneously fall by the crash factor then the value of the portfolio would equal the floor.
4. *Invest the portfolio*. Invest the portfolio in the risky assets to the level of the exposure determined in step 3 with the balance invested in the riskless asset. The riskless asset is a zero coupon bond that matures on the same day as the portfolio insurance program. For most short-dated, one year or less, insurance programs cash or short-term bills typically act as a reasonable substitute for the riskless asset. Note also that the exposure is bounded to be non-negative and typically by the value of the portfolio. The later boundary condition is to prevent gearing.
5. *Repeat steps 1 to 4*. In practice, step 4 would only be repeated if the theoretically required exposure was significantly different from the actual exposure. What defines "significantly" is typically different from one portfolio insurer to another. Some portfolio insurers have hard trading rules while others have soft rules. Some of the factors that determine whether a portfolio is rebalanced or not include the liquidity of the risky assets, transaction costs and the mandate of the insurer.

The hedging process of CPPI works in a similar way to delta hedging a call option. When the risky assets rise in value, the value of the portfolio rises and hence the value of the cushion rises. If the value of the cushion rises, the fund can afford to risk more and so the maximum exposure the portfolio can have to the risky asset rises. The opposite occurs when the risky asset falls in value. Or, put more simply, you buy when the market rises and sell when the market falls.

One important difference between CPPI and option-based portfolio insurance strategies is that if the value of the portfolio ever falls to the floor level then, under CPPI, the portfolio must be fully invested in the riskless asset. If the risky asset was then to increase in value, then the portfolio would not participate in any of this gain nor could any exposure to the risky asset be taken. Whereas, using options, some of this gain may be captured and, through delta expansion, an exposure to the risky asset may arise.

The formula for CPPI is given by:

$E = mc$

Where

E = Exposure to the risky asset

m = risk factor

= 1/crash factor

c = cushion

= $P_t - F_T(1 + i)^{T-t}$

i = annualised risk free rate

P_t = Portfolio value at time t

F_T = floor value of portfolio at maturity of the insurance program.

Exhibit 14.9 shows a typical pay-off profile of a CPPI program over a 1 year term. The impact of the insurance program is to smooth the returns relative to the same portfolio uninsured.

Exhibit 14.9

Pay-off from a CPPI Portfolio vs an Uninsured Portfolio

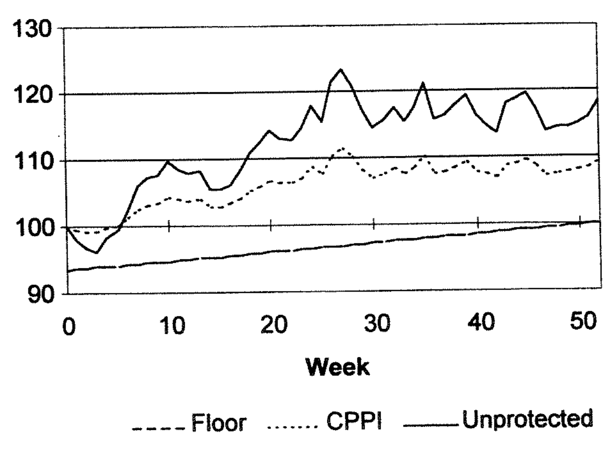
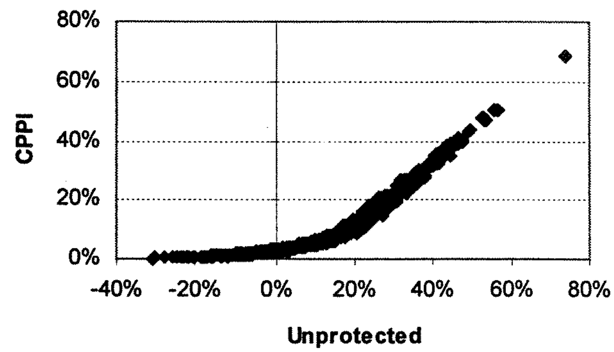


Exhibit 14.10 is a scatter diagram giving the return from a CPPI program for different returns on the same portfolio uninsured. Unlike the pay-off from options, the pay-off from CPPI is dependent upon the path the market takes over the life of the protection program. This means that for any given return

on an uninsured portfolio the return on the same portfolio insured is not fixed. *Exhibit 14.10* also shows the curved pay-off profile from CPPI.

Exhibit 14.10

CPPI vs an Uninsured Portfolio



5.1 Example 3: constant proportion portfolio insurance

Suppose an investor wanted to invest $100 in the asset described in Example 2. Furthermore, the investor wanted to ensure that the capital was insured such that in one year's time it was no less than it is today and that the investor wished to run a CPPI program to achieve this with a crash factor of 20%.

The table below shows the floor, the cushion, the exposure to the underlying and riskless assets and the value of the investment at the end of each week.

For instance, at the beginning of the first week:

Floor = 100/1.07 = 93.5

Cushion = 100 − 93.5 = 6.5

Exposure = 6.5/0.2 = 32.7

Cash = 100 − 32.7 = 67.3

At the end of the first week the portfolio's value is:

Portfolio = 32.7 * 98/100 + 67.3 * 1.07 ∧ (1/52) = 99.4

Week	Floor	Cushion	Exposure	Cash	Portfolio
0	93.5	6.5	32.7	67.3	100.0
1	93.6	5.9	29.3	70.2	99.4
2	93.7	5.4	27.2	72.0	99.1
3	93.8	5.3	26.3	72.8	99.1
4	93.9	5.8	28.9	70.8	99.7
5	94.1	6.1	30.4	69.7	100.1
6	94.2	7.0	34.8	66.3	101.2
7	94.3	8.3	41.3	61.3	102.6
8	94.4	8.6	43.0	60.0	103.0
.					
.					
.					
50	99.7	8.3	41.7	66.4	108.1
51	99.9	8.7	43.4	65.1	108.6
52	100.0	9.5	47.5	62.0	109.5

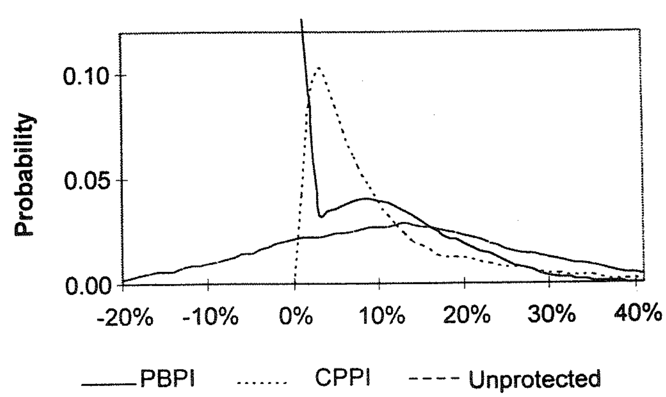
At the end of the year the investor's portfolio should be worth $109.5. Using capped, OBPI or PBPI strategies to protect the portfolio would have resulted in portfolio values of $115.3, $112.0 and $112.6 respectively.

CPPI can also be run over multiple assets or asset classes with different crash factors for each different asset or asset class. For instance, a portfolio of domestic equities, international equities and domestic bonds could be run with crash factors of 15%, 12% and a 2% rise in bond yields.

Exhibit 14.11 shows some typical distribution profiles for an uninsured portfolio, the same portfolio insured for a zero floor using CPPI and PBPI. The spike in the PBPI distribution is because all negative returns on the uninsured portfolio result in exactly a zero return insured portfolio.

Exhibit 14.11

Distribution of Returns



5.2 Example 4: a basket of options versus an option on a basket

Suppose we have a portfolio comprising $50 of asset A and $25 of asset B. The volatilities of asset A and B are 10% and 20% respectively, the correlation between asset A and B is 0.5, neither asset pays a dividend and one year interest rates are 7%. What are the value of one year at-the-money European put options on asset A and B and the portfolio/basket of asset A and B?

Asset A

$$P_{BS}(50,50,1,10\%,7\%) = 0.689$$

Asset B

$$P_{BS}(25,25,1,20\%,7\%) = 1.195$$

Portfolio A + B

Variance (A + B) = $\text{Weight}_A{}^2 * \text{Variance(A)} + \text{Weight}_B{}^2 * \text{Variance(B)} + 2 * \text{Weight}_A * \text{Weight}_B * \text{Volatility(A)} * \text{Volatility(B)} * \text{Correlation(A,B)}$

= $0.666^2 * 0.1^2 + 0.333^2 * 0.2^2 + 2 * 0.666 * 0.333 * 0.1 * 0.2 * 0.5$

= 0.01333

Volatility (A + B) = $0.01333^{0.5}$

= 11.55%

$$P_{BS}(75,75,1,11.55\%,7\%) = 1.395$$

Note that the option on the basket cost $1.395 while the basket of options costs $1.884. The basket of options cost more because there is possibility of one of the individual options expiring in-the-money while the basket option expires out-of-the-money. For example, if, at expiry, asset A was worth $40 and B was worth $35 then the put option on asset A would be worth $10 while the put option on the basket would expire worthless.

6. OTHER RISKS

Other factors to consider when considering implementing an insurance program include:

- *Credit* Are the option/futures counter parties of sufficient creditworthiness?
- *Documentation* Does the documentation cover all scenarios? How binding is the transaction?
- *Systems* How stable are the systems? What happens if they go down?
- *Accuracy* Have the systems been audited?
- *Completeness* Do the systems contain every transaction? What reconciliation procedures exist?
- *Error* What happens if there is a dealing error?
- *Liquidity* Is there sufficient liquid capital to support any funding requirements?

- *People* Are they qualified? Is it overly dependent upon key people?
- *Culture* Does a control environment exist?

7. CONCLUSION

Portfolio insurance is an investment strategy designed to meet the objectives of investors with specific market risk tolerances. However, before an investor decides to use portfolio insurance it is important that a thorough understanding of the issues involved is gained and accepted. Follow the checklist:

1. define the payoff;
2. understand the cost;
3. determine your tolerance to under performing the desired payoff;
4. appreciate when the program will fail; and
5. accept the non-market risks.

Chapter 15

Indexation of Portfolios

by Frances Cowell

1. WHAT IS INDEXATION?

Tangerines or lemons?

How do you go about choosing securities to hold in your investment portfolio? An approach taken by a Polish investor was to inscribe each of 70 tangerines with a stock code. Then he sat his pet chimpanzee ''Karolina'' down before the tangerines and recorded which ones she ate. This indicated the stocks to buy and yielded impressive results. Over a three month period Karolina chose five winning stocks, achieving a rate of return of 10%, far superior to the return over the same period achieved by a leading stockbroker.[1]

Most investors will eschew using chimpanzees, pigs' entrails or tea leaves (or at least will not admit to using such techniques). For the serious investor, the answer to this question is ''it depends''. It depends upon what information can be obtained about various stocks and bonds. The investor who is certain that a particular share is going to beat all others will logically buy as much of the stock as he or she can. The flaw in this approach is that, for the investor to be so sure, he or she must have special information, such as insider information. Access to this cannot always be guaranteed and even if it was, using it to profit from trading risks prosecution in most (but not all) jurisdictions.

Information remains the single most important factor. The job of the professional investment manager is to gain an information edge over her or his rivals. There are two main ways of doing this. The first is to find some legal way of obtaining exclusive access to the best information; the second is to make the best use possible of publicly available information. To this end, stock analysts comb through all the publicly available information they can to identify some new ''twist'' which will indicate which securities are going to do better or worse than the market. With so much brainpower devoted to working out the true value of this stock or that bond, security prices are usually at about the level they should be to reflect likely returns and risks: their ''fair price''. This makes it very difficult for the investment manager to consistently choose stocks that will give better returns than the market as a whole. Also, it is expensive.

Another approach is to buy a bit of everything in proportion to its weighting in the market. This is the essence of indexation. If your chimpanzee ate all the tangerines, you have a sort of index portfolio.

1. *The Economist*, 6 July, 1996, p 25.

If it is well-constructed, the index portfolio will rise and fall with the market it is designed to match. Because the portfolio does not seek to benefit from the relative returns to individual securities, there is no occasion to "take profits" or "cut losses". Thus the portfolio is effectively a "buy and hold": once set in place the manager can in theory pay it scant attention—although the last statement is true only up to a point.

Because indexation creates a low maintenance buy and hold portfolio, its costs compare favourably to one consisting of selected stocks.[2] Naturally this cost advantage depends on the costs of trading the component securities. For example, a portfolio of Australian equities which has a turnover of 30% will cost the portfolio from 30 to 80 basis points (hundredths of one per cent). For an international portfolio the costs are even higher, because both trading and custodian fees are significantly higher than those for domestic portfolios.

The index portfolio has another cost advantage over the actively managed portfolio: low management fees. The manager passes on to the investor the money saved from not having to conduct expensive research and analysis of the relative value of individual securities. As one would expect, management fees vary widely between markets and between managers. For an Australian equities portfolio, the difference in management fees between active and index portfolios is in the order of half of 1% per year, or 50 basis points. For an international equities portfolio the difference is about 35 basis points.

Indexed portfolios not only cost less to run than the active variety, they are an altogether less risky proposition. While both portfolios will win and lose with the market in which they are invested, the active portfolio runs the additional risk that the securities it has bought will do better or worse than the market. In contrast, the index portfolio reflects only the market performance without the additional excitement of bets placed on individual securities.

Armies of analysts find and apply the all important information which keeps securities prices fair. If all assets within a market were indexed then this information would not be used; it would be wasted, with the result that the market would be inefficient. In such a world it would be easy to achieve better than market returns using publicly available information. Indexation would then become much less popular.

The index portfolio can be sucessful only within a market which is kept efficient by the efforts of active portfolio managers.

Meanwhile, the index portfolio allows the investor something of a free ride, benefiting from all this feverish activity which keeps the market efficient, while incurring only a fraction of the costs and risks involved in achieving and maintaining this market efficiency.

Yet very few portfolios are completely indexed. Most investors recognise the need to combine active and index approaches.

2. Which must be traded frequently to benefit from the investment manager's judgments.

2. INDEXATION MEETS CAPM

To appreciate how indexation fits into an investment portfolio it is necessary to take a look at the Capital Asset Pricing Model (CAPM). This model was developed in the 1960s and is now used extensively by professional investment managers to analyse and manage the riskiness of their portfolios. CAPM says that investors receive extra returns for taking on additional risk, but only if this extra risk cannot be diversified away: taking on unnecessary (diversifiable) risk does not lead to an improvement in investment returns. CAPM can either be applied to a single asset relative to another, a group of assets within a portfolio or a portfolio of assets—relative to a nominated benchmark.

CAPM is most often applied in the context of domestic equities portfolios, but can also be applied in a more general context. CAPM helps analyse assets against some kind of benchmark. The selected benchmark could, for example, be the local sharemarket or it might be some measure of global markets. Alternatively, the benchmark could be set as a separate portfolio of assets. CAPM divides return and risk into three components which are called alpha, beta and residual risk. The return to asset *i* (r_i) is expressed as follows:

$$r_i = \text{alpha}_i + \text{beta}_i\text{i} \, . \, (r_m\text{i} - r_f) + \text{residual}_i \qquad (1)$$

Where:

r_m is the return to the market

r_f is the risk free rate of return

Thus the return and risk to an asset can be thought of as having three components:

- alpha: intentional or active risk
- beta: inherent or market risk
- residual: incidental risk or error

Past returns are not taken into account. CAPM agrees with efficient market theory that the future returns to an asset do not depend on its returns in the past. CAPM does not take into account transactions costs and other market frictions.

Alpha is the amount by which the market has underpriced asset *i*. This is what active managers seek to gain their performance advantage. Increasing the expected alpha of a portfolio will increase its expected return. If markets are efficient, alpha is equal to zero.

Beta is the sensitivity, or covariance, of asset *i* to moves in the market. The market's beta is defined as 1.0, so an asset that moves exactly in line with the market has a beta of 1.0. An asset with a beta of 1.2, for example, will overshoot market rises and falls by 20%. On the other hand, an asset with a beta of 0.9 will match only 90% of moves in both directions. A portfolio consisting entirely of cash will have a beta of zero relative to the equity market. Beta is defined as the covariance of the returns to asset_i and market_m divided by the variance of the return to the market:

$$\text{beta}_i = \text{covariance } (r_i \; r_m)/\text{variance } (r_m) \qquad (2)$$

Because beta is related to the market return, it cannot be eliminated by diversification without also eliminating the portfolio's market return. Increasing the beta of a portfolio will increase both risk and return.

The residual is that part of an asset's return that is not explained by either alpha or beta and is a random variable. The residual values of a group of assets within a period will average zero; similarly, the residual returns to a single asset will average zero over time. Adding residual risk to a portfolio will do nothing to increase its expected return, so investment managers have every incentive to eliminate this risk. They do this by diversifying the portfolio as much as practicable.

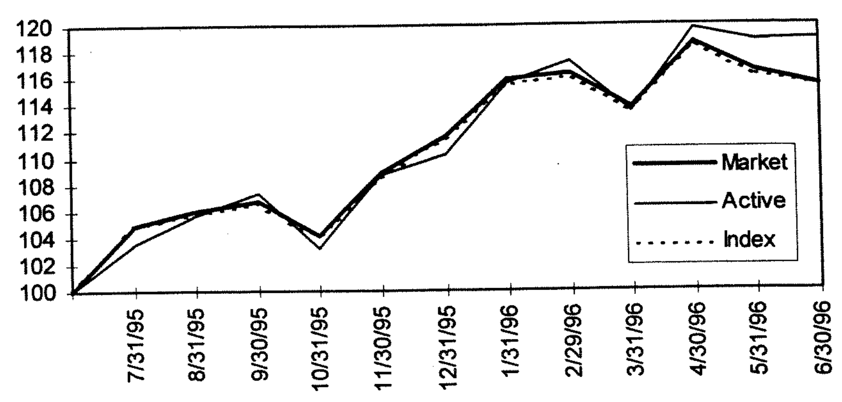
Alpha and residual risk are collectively known as specific risk because they are risks that are unique to that asset or portfolio. In the context of an index portfolio, specific risk is usually called tracking error.

The index portfolio seeks to achieve a portfolio with an alpha of zero; usually, but not always, a beta of one, and a residual as close as possible to zero, which will give zero tracking error. By contrast, the active manager seeks a positive alpha and possibly a beta which is somewhat different from one. Thus the index portfolio is a low risk portfolio with particular emphasis on low specific risk.

Exhibit 15.1 shows the difference in performance between an active and an indexed equity portfolio over a 12 month period. The distance between the the portfolios' performance and that of the market is specific risk.

Exhibit 15.1

Active and Index Performance



This low specific risk is important in the context of a portfolio invested with many managers over a range of asset classes. This is because investment managers have the resources to analyse easily the *market* risk of each of the markets in which they are invested. In contrast, effective management of the *specific* risk of each individual portolio in addition to its *market* risk is challenging for even the most sophisticated analytic tools. It follows that portfolios comprising mostly market risk with little or no specific risk ar much easier than high specific risk (active) portfolios to integrate to a complex portfolio which is invested in many markets.

This can be a powerful advantage for another reason. Research shows tha for such a diversified portfolio, 70% of total performance variation is due to asset allocation, while only 30% is attributable to stock selection. By using index portfolios the investor is able to devote greater analytic resources where these will make the most difference.

Low specific risk portfolios, such as index portfolios, resemble their benchmark. Any futures contract based on that benchmark can therefore be used as a surrogate for the index portfolio. This is particularly so in the case of equities, and can greatly facilitate liquidity management. Futures can be traded quickly and cheaply to match portfolio cash flows without affecting the portfolio balance. Futures contracts and options on futures can also facilitate the implementation of tactical asset allocation decisions, as reweighting can be effected without incurring the substantial costs of transacting physical assets. Low specific risk is also an advantage when entering into stock index arbitrage transactions (more of this later).

The low specific risk of index portfolios is useful when constructing global portfolios because physical assets can be held in the investor's domestic market and the returns swapped for those on international indices. This enables the investor to reap the benefits of dividend imputation tax credits which are lost on physical assets held offshore. It can also reduce custodian and other administrative costs.

An important feature of indexation is its cost effectiveness. Indexation can minimise transaction, management and administrative costs. Because CAPM does not accommodate these costs, index portfolios are a good instrument to use in the context of a diversified portfolio constructed on the basis of a CAPM type analysis.

Transaction costs are minimised within index portfolios because they follow a basic buy and hold strategy which gives extremely low portfolio turnover. A well constructed index portfolio can have zero turnover because the normal cash flows to the portfolio—such as dividends and coupons—are usually sufficient to facilitate periodic rebalancing as it becomes necessary. This reduces the costs of brokerage, commissions, taxes and duties on sale and purchases of securities. It also reduces the cost of market impact, which is the hidden transaction cost.

Market impact arises from adverse changes in the price of a security which are caused by the fact of trading it. It is a particular problem for very large portfolios. When trading less liquid securities, market impact can be so large that the trade becomes a self-defeating exercise.

Because the index fund does not seek to identify outperforming assets, management costs are minimised. It therefore has very low research costs, a saving which is passed on to the investor through very low management fees. Performance analysis is simplified by eliminating the need for elaborate attribution analysis, and administration costs are reduced because transactions are kept to a minimum.

Indexers are sometimes accused of naively believing that CAPM can be strictly applied in real markets. This is often unjust, as indexers are in a good position to recognise the departure of theory from practice. CAPM makes a number of assumptions that clearly do not apply in the real world: zero transactions costs is one; another is that reliable return and covariance data is always available. More importantly, although they can be difficult to identify, non-zero alphas certainly occur in the real world.[3]

Whether or not one subscribes to the efficient market hypothesis, there are several very good reasons why investors choose indexation for most, or even all, of their investment portfolios.

3. WHO USES INDEXATION?

Professional investment managers such as banks and stockbrokers, insurance companies and pension fund trustees and sponsors are the dominant users of indexation for a variety of reasons. Of course the most obvious is cost. Small portfolio investors are another example. Often these investors would prefer to invest in riskier, actively managed portfolios, but cannot because their fund is too small to meet the minimum fee scales which often apply to actively managed portfolios. If they are also unwilling or unable to invest in pooled or comingled funds such as unit trusts or mutual funds, indexation is the logical alternative.

An important and growing group of users of indexation are the large (that is, greater than US$2 billion) pension funds. To these investors, the main allure of indexation is that it is an efficient way of investing large funds, with:

- controlled risk, meaning minimum specific risk;
- minimised transaction costs; and
- minimised market impact.

Some large funds farm out management of their portfolios between several, sometimes dozens, of asset managers all operating in the same market. This leads to a further problem for these investors. This is that the use of multiple managers within a market can have the effect that the sum of all these active managers' investment decisions amounts to a very large index portfolio—but the fund is still paying active fees. This is known as a "closet index" and is clearly suboptimal.

The investor can overcome this problem by adopting what has come to be known as the "core and satellite" approach. Within any given market, the fund invests a core of 50 or 60% of its assets in one index portfolio, with the rest allocated to a small number of active managers with mandates to aggressively manage their portfolios. This approach, if effected properly, has a number of advantages:

1. It minimises the problem of market impact which otherwise would limit the investment manager's ability to assume the risks necessary to achieve acceptable returns.

3. Evidence of this is the superior returns achieved by some investment managers.

2. The active portfolios are more likely to meet their given objectives, because achieving better than market returns is much easier for small to medium portfolios than for large ones.
3. It reduces the likelihood of running a closet index because the investor has appointed a smaller number of active managers in the same market.
4. It facilitates the identification and reward of better than market performance by active managers.

Global asset managers find indexation an attractive means of gaining cost effective exposure to overseas markets. There are two ways in which indexation can improve the efficiency of global portfolio management.

The first is by simply indexing the securities within each target market. This focuses management resources on the choice of which countries to invest in. Empirical research into the performance of globally diversified portfolios supports this decision, because the results usually show that approximately 70% of the variability of returns is attributable to selection of markets, with only 30% due to security selection within markets. Logically, therefore, the global manager will devote management resources to that aspect of the fund which is likely to deliver the best results (that is, country selection and allocation). At the same time he or she eliminates the complexities of trying to manage portfolio specific risk in individual markets—often in an awkwardly different time zone. This leaves time to focus on optimising and managing the risks attributable to the markets themselves and allocating between them. Indexation offers the additional benefits of minimising management, transactions and administration costs.

The second way is to combine indexation with asset swaps. This is very useful to global managers who are subject to domestic tax. This works by arranging for a financial intermediary to swap the return on one asset, or basket of assets, for another. The global investment manager receives, over a fixed period, usually one, two or three years, the return to an agreed international asset or basket of assets. The manager pays the return to domestic assets plus a margin. The benefit to the global manager is that physical assets can be held domestically and so earn tax credits on dividends. Part of this benefit is given up in the form of the margin paid to the intermediary, but part is retained by the fund. Structured correctly, this can be a most efficient means of global investing for a taxable investor.

Exhibit 15.2

Asset Swap Between Investors in Markets A and B

Investor A		**Investor B**
+ Return on Market A	*Earned on index portfolio of physical assets held*	+ Return on Market B
– Return on Market A	*Swapped asset returns*	– Return on Market B
+ Return on Market B	*Swapped asset returns*	+ Return on Market A
+ Imputation tax credits A	*Earned on index portfolio of physical assets held*	+ Imputation tax credits A
– Margin		– Margin
+ Return to Market B **+ Imputation tax credits A** **– Margin**	***Net Outcome***	**+ Return to Market A** **+ Imputation tax credits B** **– Margin**

Exhibit 15.2 shows an asset swap between investors A and B, each of whom pay tax in their home market but wish exposure to the other. Both parties hold physical assets domestically. Their net outcome is the foreign asset return plus imputation credits in their home market less a margin paid to the intermediary for arranging the swap.

It is usual for the intermediary to seek a taxable investor in another country who can hold physical assets in her or his home market but wishes exposure in another. While this is preferable because the risk incurred in the original deal can be wholly or partially neutralised, it may not always be possible to arrange. Should this be the case, and it often is, the intermediary will be obliged to establish an index portfolio to offset her or his exposure. Thus another group of indexers becomes apparent: the financial intermediary who arranges the asset swap with the global manager.

Another group of indexers seek risk free returns using index portfolios in conjunction with derivative instruments such as share price index futures contracts. Often referred to as arbitrageurs, these indexers usually work for large stockbroking houses and may purchase the index portfolio on behalf of the house or for a client. They differ from most other indexers in that they usually work on a very short time horizon.[4] The opportunities to derive extra-market or arbitrage profits through trading mispriced futures and options contracts tend to be extremely short-lived, rarely more than a few minutes. So, while pension fund trustees and sponsors hold their index portfolios for many years and investment bankers hold theirs for anything from one to three years, the arbitrageur will often buy and sell an index portfolio within a few hours. This time element has important implications for the construction and implementation of the index portfolio.

4. Arbitrage opportunities depend upon volatility in the price of the derivative relative to that of the underlying physical assets. Volatility in "relative price" or premium is usually associated with volatility in the underlying market.

4. HOW MUCH INDEXATION GOES ON?

The main group of indexers are professional investors responsible for managing large funds over extended periods of time, such as pension plan sponsors and trustees, mutual funds and insurance companies. Taken as a group, these institutions invest approximately 20% of their assets in index portfolios of one sort or another, mostly equities. Indexation has proved most popular with long-term investors in the United States, where nearly 30% of these funds' investments are indexed. In the United Kingdom the figure is closer to 20%. There is as yet little interest in indexation in continental Europe or in Asia. In Australia and New Zealand the proportion indexed is probably just under 15%.

Indexation became popular in the United States in the late 1970s and early 1980s. Interest was first raised by early research into the relative performance of equities managers which indicated that few, if any, active managers consistently did better than the S&P500—the market in which they invested. While each measurement period saw individual managers outperform, it was rare for better than market performance to occur over extended periods under any individual manager. More importantly, from the point of view of the investor, performance was very difficult to predict.[5] Needless to say, the picture was not improved when transactions costs and management fees were taken into account. By early 1987 approximately 9.5% of pension fund assets in the United States were indexed. The following year saw this figure grow to about 11%.[6]

So how much will indexation grow? In the first part of this chapter we found that there is a limit to the amount of indexation that can happen: if all portfolios were indexed, then the market would lose the efficiency upon which indexation relies for its success. So far nobody has succeeded in quantifying this limit. Early guesses put the figure at about 20% but this has already been exceeded in the United States, and is therefore almost certainly an underestimate. The answer will depend in part on the characteristics of the market in question. In particular it will depend on the relative importance in the market of the investors, such as those appearing in section three, who have most to gain from indexation.

For example, a market which is dominated to an unusual degree by very large funds will see more indexation than otherwise, because these funds are likely to favour the passive core and active satellite approach to managing their assets. Similarly, a market with a significant representation by offshore investors will see more indexation, either directly through portfolio share holdings or indirectly through counterparties to asset swaps.

On the other hand, the natural limit to the amount of indexation a market can support will be determined by the effectiveness of the market efficiency counterfoil. A market with even a small number of very efficient and competitive security analysts and traders will support more indexation than

5. A recent study by Lakonishok, Schleifer and Vishny examines this phenomenon in the US equity market over the period 1983 through 1989. Josef, Lakonishok, Andre, Schleifer and Robert W Vishny "Study of the US Equity Money Manager Performance," Brookings Institute Study, 1992.
6. *The Economist*, 27 August, 1988, p 63.

one where most investors are less aggressive. The former will do more to maintain market efficiency than the latter, who may allow some securities to remain mispriced.

Despite the limits to the amount of indexation within markets, there remains room for indexation to grow as a proportion of total investments. This growth will come from four main sources.

The first of these is pressure on costs. One of the dominant features of the managed funds industry, in most mature markets, is the downward pressure on the general level of management fees and other investment-related costs. This pressure has increased significantly over the decade to the mid-1990s and has a number of sources.

1. The most important of these is increased competition in the managed funds market. While a decade ago this market consisted mostly of life insurers and corporate pension funds, whose clientele were "captive" to some extent, today the range of participants is considerably wider. Evidence of this is that on both the London and New York stock exchanges, the number of listed equity trusts now exceeds that of listed equities. Increased competition is the result of, among other things, a trend in mature markets to the deregulation of financial systems. Deregulation has helped remove many of the barriers which hitherto kept many would-be investment managers from setting up shop.

As the size, in real terms, of managed funds has grown, so management fees have been squeezed. As management fees are usually levied as a percentage of the value of assets under management, the higher the absolute amount under management, the lower the fee in percentage points required to cover the manager's costs.

This real growth has three sources:

(i) deregulation and increased competition: lower fees to managed funds attracts more investment, and so on (this might be called a virtuous circle);

(ii) an increased saving rate in some rich countries, whose populations are entering the period of high savings which precedes the retirement of a large and affluent section of the population; and

(iii) real market returns are mostly reinvested in managed funds, so as markets for financial assets deliver positive real rates of return, the market will continue to grow by the real rate of return.

2. Increased competition in the market for managed funds has necessitated the improvement of capabilities in evaluating the effectiveness of asset managers. This increased efficiency has generally allowed investors to bid down the fees charged by asset managers, although in some spectacular cases the opposite has happened.

3. The early 1990s has seen a trend in most OECD countries toward lower nominal rates of return for most asset classes, mostly driven by lower inflation rates. Investment returns achieved by most asset managers are quoted both before and after fees. The cost factor represented by management fees are more obvious when compared to nominal returns which are low, than to those that are high.

Another source of growth in indexation is the increased use of the core-satellite approach associated with the growth of very large funds. This is partly a function of the growth in the market for managed assets, but it is also determined by how that growth is manifest. Where its main source is increased retirement saving, it is likely that these assets are managed within a small number of very large funds. This effect is, of course, most obvious where the primary vehicle for retirement saving is a public pension plan, but similar effects can be seen where saving is mostly through private pension schemes.

Schemes which are large relative to the market in which they invest may choose the index core-active satellite approach to assume appropriate levels of market risk while avoiding costly market impact. This avoids ''closet indexation'' and facilitates the identification and reward of active managers who deliver better than market returns.

Market concentration can also contribute to indexation. A recent example of this is given by the merger of CIBA-Geigy AG and Sandoz AG to form Novartis AG. This merger has caused the highly concentrated Swiss equity market to become even more so: together with Roche AG and Nestle SA, the three companies comprise nearly 60% of the Swiss Market Index. This heavy concentration led the Canton of Zurich pension fund to decide to boost passive management from 40% to 85% of its domestic equity allocation.[7]

Increased globalisation is contributing to the growth in indexation because global investors are, on balance, more likely to index some of their offshore assets than are local investors. Global investors are apt to choose indexation because it enables them to focus management resources on country and asset class selection. It is in this area, rather than in intra-market security selection, where they have most competitive advantage. Indexation can also save on global transactions and administration charges.

Investors are choosing to invest increasing proportions of their assets outside their home countries for some very good reasons. Investing outside one's home country can vastly improve the risk-return profile of a portfolio because it increases diversification. Until recently, international portfolio investing was limited in many rich countries by exchange controls and other financial regulations, high transactions costs and the limited ability of many investors to adequately analyse and control the risks of international investing. For investors in most developed markets these restrictions are rapidly falling away, with the result that investors are casting their nets further and further afield in search of returns and diversification not available in their home markets.

The most significant growth in offshore portfolio investing has happened in the United States. As recently as 1989 the proportion of US portfolio investments held internationally was less than 4%; in 1984 it was less than 2%.[8] By the early nineteen nineties the figure was closer to 20%. In contrast, portfolio investors in the United Kingdom are accustomed to investing well over 20% of their assets offshore. International diversification is likely to continue to grow for the following reasons.

7. *Pensions & Investments*, 8 July, 1996, p 14.
8. InterSec Research Corp, quoted by Institutional Investor, February 1991, p 97.

- Investors in mature markets in continental Europe will continue to increase international diversification as further deregulation permits, and as returns in their home markets increasingly fail to satisfy their demand for better risk adjusted returns.
- Similarly, investors in newly-emerged markets in Asia and elsewhere will seek increased international diversification as a result not only of deregulation but also because the size of investment funds is already increasing dramatically, relative to investment opportunities at home.
- Emerging markets in south and south-east Asia, Central and South America, Eastern Europe and, in the longer term, Africa, will become increasingly attractive to portfolio investors of all origins seeking to maximise returns and diversification.

As this trend gathers pace, indexation will follow in its wake. Associated with the growth in international diversification of investment portfolios is the growth in the market for asset swaps. The two trends will continue to be closely related as long as there exists a tax advantage to the international investor in using swaps over physical equities or other related instruments.

Both will continue to help fuel the worldwide growth in indexation.

5. APPROACHES TO INDEXATION

The steps to setting up an index portfolio are to:

(a) define the benchmark;

(b) set target level of tracking error;

(c) determine type of portfolio construction;

(d) decide rebalancing rules; and

(e) plan implementation.

(a) Defining the *benchmark* is not always as simple as it sounds. Most investment managers are acutely aware that, whatever the stated benchmark, the real benchmark is the performance achieved by their competitors.

Nevertheless, a formalised benchmark is a necessary part of any investment management mandate because it provides the investor with a means of objectively assessing the performance of the investment managers who have been appointed (and those who have not). Features which are necessary and or desirable for a benchmark are:

- It must meet the investment objectives of the investor. Usually this means that it must give a broad coverage of the market in which it will invest. In some instances this may necessitate designing a customised benchmark either within a recognised asset class or as a composition of different asset classes or parts of asset classes.
- It should be investable. In other words the securities that make up the benchmark should be freely traded on a recognised exchange.
- Derivatives are a big help. For the purposes of liquidity management and periodic asset class reweighting, there is an enormous advantage in selecting a benchmark on which futures contracts are traded. This is not

always possible even for domestic equities portfolios, and is not generally available for international asset classes.

- Public quotation reduces ambiguity. While it is preferable to identify a benchmark which is quoted publicly, customised or less widely recognised benchmarks can work well provided the components are publicly quoted. This allows independent computation of benchmark performance by investor, manager and custodian; so avoiding confusion about the relative performance of the portfolio.

(b) The next step is to decide how much *tracking error* the portfolio can tolerate. This will be determined by the size and purpose of the portfolio and how difficult and/or costly it may be to achieve close to zero specific risk in the portfolio.

Ideally, the indexer will aim for zero tracking error which, combined with a beta of exactly one will give perfect index returns. But this reckons without transactions and other costs, which can have a significant effect on the outcome. In some cases the investor is better off with some tracking error because this flexibility can help reduce the other costs of the portfolio.

(c) Having determined the benchmark and its tolerance for tracking error, the indexer must now decide the best way to go about *portfolio construction*. The inputs to this part of the indexation process are:

- the benchmark;
- the tolerance for tracking error;
- the purpose and expected life of the index portfolio—long-term or short-term;
- the form of the existing portfolio—cash, shares, et cetera.
- estimates of transactions costs for each asset in the sample; and
- estimates of beta values for each asset in the sample.

The first part of this process is to decide whether to fully replicate the benchmark index or to adopt a sampling approach, and, if the latter, how many securities to include in the sample. The names provide apt descriptions: full replication is the process of simply buying every security in precise index proportions, while sampling is where only some securities are bought.

The choice is usually determined by the portfolio's tolerance for tracking error and the structure of the benchmark. If it is made up of a great number of securities then sampling is almost certainly the right approach. If there are only a few dozen securities in it then full replication could be the answer, provided each component is sufficiently liquid and trading costs are not too high.

Full replication will give portfolio performance which is very close to, but not identical with, the benchmark. The difference stems from the fact that all benchmarks change their components from time to time and the replicating indexer must follow suit. This results in trading costs which, together with administrative costs, have a negative impact on performance. Sampling incurs lower rebalancing and administrative costs (because there are fewer securities to trade and changes in the benchmark can be followed less rigidly), but incurs a larger tracking error. Tracking error can be positive or negative. The objective of the sampling indexer is to minimise this error, so delivering portfolio performance with variations from the benchmark which are not only small, but are positive as often as they are negative.

There are a number of approaches to sampling, including random sampling—with or without the assistance of Karolina the chimpanzee. We will concentrate on only two—stratified sampling and optimised sampling—which are often applied in tandem.

Stratified sampling is where the benchmark is broken up into bite-sized chunks and securities selected from each chunk to make up a portfolio. In the case of a domestic equities portfolio, the bite-sized chunks might be determined by industry groupings, with the result that each industry is given proportional representation in the sample portfolio. An international equities index might start with a country by country approach. A property portfolio might seek to separate the index into different property sectors, such as commercial, industrial and residential. For fixed interest one might approach the task by looking at various credit exposures within the index.

Example of a stratified sample

The following example illustrates how a stratified sample might work. The indexer wishes to construct a 100 stock portfolio to track a benchmark which itself has over 300 stocks. The simplest approach would be simply to select the 100 largest stocks by market capitalisation. For the sake of the example this portfolio will be called *TOP100.*

The indexer then notices that the benchmark is divided into 24 industry categories. Since it makes sense to at least try to match industry weightings to the benchmark,[9] the indexer decides to select 100 stocks according to industry categories.

The first step is to decide which stocks to select. Obviously the largest stock in each industry group will be included. (If the number of industry groups exceeded the number of stocks, that is, there were more than 100 industry groups, then one stock would be selected from each of the 100 largest industry groups by market capitalisation.) The next step will select the next largest stock from each industry group.[10]

Next, the indexer must weight the stocks. Industry groups will be weighted according to their weight in the benchmark index. Within industry groups, stocks will be weighted according to their relative weight within the industry. Individual stock weightings will therefore vary from their weight in the benchmark index.

The 100 stock portfolio selected by this process is called *SAMP100.* *Exhibit 15.3* shows the composition of the benchmark and the two portfolios.

9. The usefulness of matching industry weightings depends on the relevance of the industry group definitions to the activities of companies in the benchmark. In many cases companies are arbitrarily assigned to whichever industry group represents the largest of its business lines, despite the fact that, in some cases this may contribute less than half of that company's overall profit.
10. The indexer may omit some stocks because they are unacceptably illiquid. The result will be that some industry groups will have a larger number of stocks in them than others.

Exhibit 15.3

Stratified Samples

Industry Group	*Benchmark Weight*	*No of Stocks*	*TOP100 Weight*	*No of Stocks*	*SAMP100 Weight*	*No of Stocks*	*SAMP112 Weight*	*No of Stocks*
Gold	4.83%	43	3.76%	10	4.84%	9	4.09%	10
Other Metals	6.44%	19	6.14%	6	6.44%	6	6.68%	8
Diversified Resources	14.24%	4	16.69%	3	14.24%	3	15.10%	3
Energy	4.19%	14	4.27%	6	4.21%	6	4.19%	6
Infrastructure & Utilities	0.96%	6	0.94%	2	0.96%	1	0.73%	1
Developers	2.97%	12	3.01%	3	2.97%	3	3.03%	3
Building Materials	4.25%	11	4.64%	5	4.24%	6	4.33%	5
Alcohol & Tobacco	2.05%	5	2.20%	3	2.05%	3	2.18%	3
Food	3.37%	8	3.43%	3	3.37%	4	3.49%	4
Chemicals	1.41%	5	1.57%	2	1.42%	1	1.53%	2
Engineering	0.75%	8	0.32%	1	0.74%	1	0.58%	2
Paper & Packaging	2.15%	5	2.39%	3	2.15%	3	2.33%	3
Retail	3.41%	12	3.43%	4	3.42%	5	3.40%	3
Transport	3.07%	11	3.41%	3	3.08%	3	3.24%	3
Media	8.47%	19	9.09%	7	8.47%	5	8.62%	8
Banks	18.61%	14	21.62%	10	18.60%	10	18.91%	9
Insurance	2.31%	9	2.37%	5	2.32%	2	1.88%	4
Telecommunications	0.23%	6	0.00%	0	0.23%	1	0.00%	0
Investment Services	2.07%	22	1.11%	3	2.06%	3	2.52%	7
Property Trusts	4.51%	41	2.67%	5	4.49%	7	4.41%	9
Miscellaneous Services	1.38%	26	0.27%	1	1.38%	3	0.82%	3
Miscellaneous Industrials	1.42%	15	0.59%	2	1.42%	4	1.54%	6
Diversified Industrials	3.96%	12	3.77%	6	3.96%	6	3.96%	4
Tourism	2.95%	21	2.32%	7	2.97%	5	2.44%	6
Total	*100%*	*348*	*100%*	*100*	*100%*	*100*	*100%*	*112*

The *TOP100* portfolio is heavily weighted in diversified resources, media and banks, but has no exposure at all to telecommunications. *SAMP100* on the other hand, matches its industry group weightings to the benchmark for each industry group.

Stratified sampling has intuitive appeal and will result in satisfactory risk control so long as the categories are chosen well. If categories are ill-chosen then risk control is liable to fail. Another, often more robust approach is called optimisation, sometimes known as mean-variance optimisation.[11] *Optimisation* seeks the lowest tracking error for the index portfolio by drawing on historical data to maximise diversification relative to the benchmark. To do this it takes into account the historical correlations between securities and groups of securities. From this it constructs a portfolio which will give the lowest expected tracking error. It does this iteratively, in other words it builds the portfolio one "trade" at a time—measuring the expected tracking error of the portfolio after each "trade" until the portfolio

11. For general (active) portfolio construction, optimisation seeks to identify the lowest levels of risk for varying amounts of expected return.

is complete and there is little or no benefit to be derived from further tinkering or iterations.

There are a number of approaches to optimisation, including the linear and quadratic varieties, with a healthy debate raging about the relative merits of each. The two most popular are *factor* models and *industry mean* models. Both are based on the main principle of CAPM, that while future asset returns cannot be derived from past returns, the risk characteristics of an asset (and therefore of a portfolio of assets) tend to be stable over time.

Put simply, the factor model uses either a combination of correlation analysis and intuitive interpretation, or a cross-sectional regression analysis of past security returns. The purpose is to identify factors which are common to groups of securities in the sample, and to quantify the relationship between the factors. It then groups each security in the sample according to its main factor, and tries to achieve a portfolio factor exposure which is similar to the benchmark.

The industry mean model, on the other hand, groups securities according to their main industry and tries to quantify the correlation between these groups. The aim is to give a portfolio which is diversified according to the observed correlation between main industry groups.

To show how optimisation can affect a portfolio's expected risk, the two portfolios above were optimised against the 348 stock benchmark. The optimiser was allowed to change the weightings of each stock but was constrained to the same sample of stocks. Then the optimiser was allowed to add 12 stocks to the *SAMP100* portfolio to give a portfolio called *SAMP112*. The expected tracking error of each portfolio is set out in *Exhibit 15.4*.

Exhibit 15.4

Tracking Error—Stratified Sample vs Optimised

	TOP100	*SAMP100*	*SAMP100*	*SAMP112*
	Stratified	Stratified	Optimised	Optimised
Expected Tracking Error	0.68%	0.56%	0.42%	0.33%

These results show that matching industry weightings to benchmark gives lower expected tracking error, that this result can be improved by optimising the portfolio, and improved even further if extra stocks are added in the optimisation step.

SAMP112 in the first table shows that an optimised portfolio does not necessarily match industry weights to benchmark, although divergences tend to be smaller than those for the *TOP100* portfolio. The optimisation process takes into account correlations between stocks and industry groups to describe a portfolio with the best overall diversification across assets.

As one might expect, the process of portfolio optimisation requires substantial computer resources. For this reason optimisers have been widely used in portfolio construction only for the last decade or so. An endearing feature of optimisers is that they, like all computer models, are subject to the GIGO principle—Garbage In Garbage Out. The optimising indexer must therefore be able to judge if the portfolio defined by the optimiser meets its objectives. A capricious charm of the optimiser is that it can give results which appear much more reasonable than they are. Contrary to the laws of probability, and a sop to the technophobe, the resulting error seems always to be both negative and large.

The combination of stratified sampling and optimisation can be powerful if the benchmark is amenable to stratification, the strata are well chosen, and there exists historical data in sufficient quantity and quality to enable a meaningful optimisation.

There is no single approach to indexation which is always better than others. The successful index portfolio is one that is designed to fit the requirements of the individual mandate even if it requires a customised benchmark. The market, the benchmark, the investment objectives and constraints in the mandate will determine which is the best approach for a given indexed portfolio.

(d) *Rebalancing* rules should be set next. How often this happens will depend on the level of transactions costs and the amount of specific risk that can be tolerated: high risk tolerance allows less frequent rebalancing because the cost of a rebalance will be justified less often by the expected reduction in specific risk. On the other hand, low risk tolerance requires more frequent rebalances. Rebalancing will also be influenced by the frequency and timing of cash flows to the portfolio. If rebalancing can be timed to coincide with cash flows either in or out of the portfolio, then rebalancing costs can be sharply reduced.

(e) *Implementation*: the objective is to minimise transactions costs and execution risk.

The best implementation method depends partly on what shape the existing portfolio has. If for example, it is made up entirely of liquid assets and the relevant share price index futures contract is trading below its fair value, then the indexer can add value by buying futures. These can then be exchanged for physical stock when the futures price is at fair value or higher. However, it is usual for the starting portfolio to comprise an active or indexed portfolio of physical shares and liquid securities. The indexer must then calculate the volume of shares to be sold and bought. If this is relatively small and the trade is made up of a small number of large parcels of stock, then it is most efficiently implemented by simply placing normal buy and sell orders with brokers.

When confronted with the necessity of trading a large volume of stock, however, the indexer must take care to manage the execution cost. In some markets this can be substantial enough to alter the performance of the portfolio or the profitability of the arbitrage. Execution cost consists of a number of elements:

- taxes;
- commissions;
- bid-ask spread;

- market impact; and
- opportunity cost.

The first two of these are usually known or can be accurately estimated in advance. The bid-ask spread is the difference in the prices bid and offered in the market, either by market makers or by other investors. Generally, but not always, the smaller the stock the larger the bid-ask spread.

Market impact can be thought of as the cost of transacting each additional share. For example, consider the buyer of one share; he or she will accept the offer price which will probably remain unchanged after that transaction. If the same investor buys one thousand of the same shares the price may move slightly after the trade is complete, as each marginal seller in turn has completed her or his order. A buyer of one million of the same shares may find that the sale price (and sometimes the bid price) moves even before the trade is complete, as other market participants read the signal that demand for the stock—and therefore its implicit value—has increased sharply.

Opportunity cost applies more to active portfolio managers who have more discretion over the timing of their market activities than do indexers, who are largely bound to follow predefined decision rules. It is the cost incurred when an investor sets out to trade a stock at a certain price and fails to complete the trade because the stock price keeps moving beyond the limit set.

Execution cost and risk can be controlled by executing a "block trade". This is the practice of buying (or selling) the entire portfolio as one parcel of stock. This can reduce the execution risk on the portfolio to zero by transferring this risk to the stockbroker who will effect the transaction. It also reduces the likelihood of dealing errors and can partially streamline the consequent paper trail. The attractiveness of this strategy depends, of course, on how much it costs. This will in turn depend on:

- which stocks are to be traded and in what proportions;
- the amount of liquidity in the market at the time of the trade; and
- the willingness of brokers to execute the trade.

While the portfolio manager has little control over the first two variables, it is possible and desirable to influence the third. The best way of doing this is to seek competing bids from more than one broker. This helps ensure that the price obtained for the transaction is competitive, because the broker with the greatest appetite for the trade will bid the keenest price. It will reflect both the price, including any taxes, at which the broker can buy or sell each stock, either in the market or against existing client business, and some charge to compensate for the risk of taking positions in some of the stocks as principal. The magnitude of this charge will depend on how long the broker expects to have to hold the position, interest rates and other funding costs and hedging costs. A basket of stock which is easy to either trade or hedge in the market will attract a smaller principal charge.

Similar principles will direct the strategy employed to rebalance the portfolio from time to time, although the effect on the portfolio will be smaller.

6. CASE STUDY: DOMESTIC AND INTERNATIONAL INDEXES

The following hypothetical example illustrates how indexation fits into a diversified institutional portfolio.

A medium-sized corporate pension fund, based in Australia, seeks to construct a portfolio that will give optimal long-term gains while controlling risk and cost in the short term. The fund must meet regular redemptions, so it cannot tolerate short term fluctuations in the value of its investments. On the other hand, it also has a regular flow of new funds with a significant quotient of young members with long-term investment needs. The adviser has recommended that about 10% of the overall fund be invested in fairly liquid securities. This will conservatively meet all likely liquidity requirements and allow periodic rebalancing.

The fund's adviser has recommended a mix of growth and defensive assets with a strong emphasis on risk diversification. This is designed to give participation in world growth with controlled volatility. The domestic equity portfolio should be invested mostly in a broad-based share index with a small but significant proportion of the domestic equities portfolio invested in small capitalisation stocks. For the international equities portfolio, both developed and emerging markets should be represented according to their weighting by capitalisation in world equity markets overall.

The fund will also have some allocation to domestic property markets through listed property vehicles. It is recognised that this is not a good substitute for direct property holdings, but the fund size and limited liquidity preclude direct investment in the property market.

Domestic and international fixed interest provide a diversified "defensive" element to the portfolio. Fixed interest markets are usually less volatile than equities and can provide valuable diversification to the overall portfolio.

The pension fund sponsor is aware that transactions costs have in the past contributed significantly to the underperformance of the fund and so is keen to do all that is possible to keep these to a minimum. For this reason the allocation between asset classes is to be held constant. The portfolio will be rebalanced frequently back to the original asset allocation using natural cash flows.

The sponsor therefore determines seven indexing mandates:

(i) domestic equities—large capitalisation;

(ii) domestic equities—small capitalisation;

(iii) international equities—developed markets;

(iv) international equities—emerging markets;

(v) domestic listed property securities;

(vi) domestic fixed interest; and

(vii) international fixed interest.

These mandates cover each asset category except cash. Each has a different benchmark and is subject to different market conditions

(i) *Domestic equities large capitalisation* is benchmarked to a broad-based local share price index. In the United States market, for example, the most popular benchmark is the S&P500 index. Our investor has settled on the Australian All Ordinaries Accumulation Index, which satisfies the main requirements of being investable, publicly quoted—in the form of both price and accumulation indices—with a lively futures and options market. The All Ordinaries has some fairly serious shortcomings which are common to many equity benchmarks. That is, it omits some serious sectors of the economy as a whole. The All Ordinaries is designed to capture 90% by capitalisation of all stocks listed on the Australian Stock Exchange, so it misses the smallest 10% of listed securities. More importantly, it misses all unlisted securities[12]—a significant part of the real economy which includes nearly all transport infrastructure, most power generation and most medical services—to name some examples.

(ii) *Domestic equities small capitalisation* is benchmarked to the Australian All Ordinaries Accumulation Index excluding the top 100 stocks. This benchmark is broad-based and publicly quoted but does not have a futures market.

(iii) *International equities developed markets* are benchmarked to the Morgan Stanley Capital International World Index. This benchmark is publicly quoted and has a broad coverage of developed markets. While the MSCI world index lacks a futures market, most of its component countries have actively traded share price index futures contracts.

(iv) *International equities emerging markets* are benchmarked to the Morgan Stanley Capital International Emerging Markets Index. This benchmark, like the domestic equities small cap benchmark, is broad-based and publicly quoted but does not have a futures market.

(v) *Domestic listed property securities* are benchmarked to the Australian Stock Exchange Listed Property Accumulation Index. Again, this benchmark is broadly representative of its sector and is publicly quoted, but does not have any associated derivatives markets.

(vi) *Domestic fixed interest* is benchmarked to the SBC Warburg Semi-Government All Maturities bond index. This benchmark is one of several widely recognised fixed interest indices and its component stocks are publicly available and tradeable. It does not have an associated derivatives market, but futures and options contracts are available on ten and three year Commonwealth Government bonds, which can be used as a surrogate for the purposes of liquidity management and for matching duration.

(vii) *International fixed interest* is benchmarked to the Salomon Brothers World Government Bond Index. This benchmark is widely recognised, with components publicly available and generally tradeable. No single derivatives market covers this index but many component countries have one or more futures contracts on government bonds which can be used as an approximation to the index.

For each international portfolio, currency is to be unhedged. In other words, the portfolio performance will be determined not only by the

12. These sectors are held either in unlisted private companies, offshore equity owners or they are publicly owned.

performance of domestic and international stock and bond markets, but also by currency movements.

In recent years index enhancement has gained popularity. Index enhancement seeks to achieve better than market or index rates of return, by adding something to the basic indexing approach. The subject of index enhancement is dealt with in a later section of this chapter. For our example, all of the indexed sectors will avoid risky enhancements.

7. EQUITY INDEX PORTFOLIOS

Following the example set out above, we have four equity index portfolios with four benchmarks:

Exhibit 15.5

Benchmarks

Domestic equity—large cap	Australian All Ordinaries Accumulation Index
Domestic equity—small cap	Australian All Ordinaries Accumulation Index ex the Top 100 stocks
International equity—developed markets	Morgan Stanley Capital International World Index
International equity—emerging markets	Morgan Stanley Capital Index Emerging Markets Index

Each portfolio will require its own approach while the general principles of index portfolio construction are applied. This will achieve the twin aims of the indexer, which are the minimisation of cost and risk.

The *domestic equity—large capitalisation* portfolio is best managed using the stratified sampling with optimisation approach already described. Stratified sampling is favoured over full replication because the benchmark index contains well over 300 stocks, many of which trade very rarely. Optimisation can reduce the expected tracking error because there exists sufficient reliable historical stock data to build a useful correlation matrix. From this matrix the diversifying effects of individual stocks can be estimated. This structure will give a portfolio with about 150 stocks in it. It will have an expected tracking error of about 0.25% pa.

The number of stocks selected for the portfolio is important. The index is devised to represent 90%, by market capitalisation, of Australian listed equities: it comprises between 340 and 350 stocks. Too many stocks in the indexed portfolio will increase the transactions and administrative costs of the portfolio, while too few stocks will subject the portfolio to excessive tracking error. The balance between the number of stocks held, tracking performance and estimated rebalancing costs is set out in *Exhibit 15.6*.

Exhibit 15.6

Number of Stocks vs Tracking Error

Number of Stocks	Per cent of Benchmark by Capitalisation	Estimated Rebalancing Costs	Estimated Tracking Error
70	79%	0.40%	0.55%
100	85%	0.45%	0.42%
150	92%	0.69%	0.19%
180	95%	0.86%	0.14%
250	98%	1.05%	0.05%
300	99%	1.22%	0.01%

This shows that reducing the tracking error from 19 basis points to five basis points by increasing the number of stocks from 150 to 250 increases annual rebalancing costs by 36 basis points. In this case the indexer is clearly better off with fewer stocks.

In general, portfolios with more stocks in them will require rebalancing more often because they are more subject to small changes in the benchmark index which can be tolerated by a sample portfolio with higher tracking error tolerance. On the other hand, reducing the number of stocks to 100 can increase the tracking error of the portfolio to approximately 40 basis points, which exceeds the portfolio's specific risk tolerance of 0.25%.

In the case of Australian equities, the choice of the number of stocks held can also have an impact on the beta of the portfolio. This is because, unlike many other equity markets, large Australian stocks tend to have a beta greater than one. Because of this, the beta effect will often show up as a size bias in the portfolio.

An efficient rebalancing strategy is critical to the success of the portfolio.

With about 150 stocks, cash flow to the portfolio needs to be only 3% or 4% per year to allow adequate rebalancing without having to sell any shares. This means that the effective portfolio turnover is close to zero, so transactions costs are very low indeed.

It also means that from time to time the cash balance in the portfolio will be as much as 5%. This liquidity should be ''equitised'' by buying the appropriate number of futures contracts.

To calculate the number of futures to give equivalent exposure to the equity market, the following algorithm is used:

Number of Contracts = Face Value/(Index Level . Point Value of Futures) (3)

For example, if the investor has \$1 million of cash to equitise, the current market level is 2202 and the point value of the futures contract is \$25, the number of contracts to be purchased is calculated as:

Number of Contracts = \$1,000,000/(2202 . \$25)
= 18

Buying futures ensures that the cash component of the portfolio participates in the performance of the equity market. When a rebalance is deemed necessary, futures contracts are sold and physical shares are purchased.

When the portfolio has been optimised and the indexer has determined the portfolio's composition, it remains to be determined how best to purchase the required stocks. The choices will generally include purchasing physical stocks or buying futures contracts which can later be exchanged for physical shares.

Purchasing physical shares often simply means placing an order to buy each of the component assets, but sometimes the indexer may choose to buy the portfolio as a block trade, as described in an earlier section. In some conditions this form of trade can be very cost effective for the indexer, but only if he or she has the capability to accurately evaluate the cost of the trade in comparison to normal on-market transactions.

The other way of implementing an index portfolio is to substitute futures contracts for all or part of the portfolio. This strategy is normally employed for a large trade where the futures market offers better liquidity than the underlying physical assets. It can also happen that using futures contracts can add risk-free returns to the portfolio, which would not be available by a straightforward purchase of physical shares. This type of trade is, strictly speaking, an enhancement, and is dealt with later in the chapter.

The *domestic equity—small capitalisation* portfolio would ideally be constructed following the same principles as the large capitalisation portfolio. The difficulty is that optimisation is precluded by the lack of adequate historical return data—and correlation matrix—for the stocks which make up the benchmark. The indexer must therefore make do with a stratified sample. A portfolio of 100 stocks will cover 73% of the capitalisation of this market. Adding 50 stocks brings this coverage to 88%.

The chosen method of implementation is likely to be limited to ordinary buy and sell orders. Execution may take days or weeks for some stocks.

Both tracking error and rebalancing costs will be higher for this portfolio than for the large capitalisation portfolio. In addition, liquidity management will be complicated by the fact that the only available derivatives contract offers at best an imperfect hedge against the small capitalisation benchmark.

International equities—developed markets: this index portfolio requires a totally different approach to those adopted for domestic portfolios. One reason for this is that the benchmark MSCI index comprises over 1,500 stocks, rendering it unlikley that it would be a full replication portfolio. Another reason is that the indexer is likely to treat each component country as a separate portfolio, mainly because, while the technology exists to optimise a global equities portfolio across borders—representing a more efficient solution in terms of risk diversification—it is still fairly new and requires considerable computing power. Most portfolio construction software therefore makes do with a country by country analysis. Given this approach, the indexer of international assets is much more likely to employ derivatives markets than is the domestic indexer. This is mainly because of economies of scale and transactions costs, but also because the logistics of constructing and maintaining optimised portfolios of physical shares within each country demands a more streamlined approach.

The main approaches to indexing international equities include

- buy physical shares in optimised portfolios within each country, with country weightings matching the benchmark index;
- buy share price index futures in each country, with country weightings matching the benchmark index;
- hold domestic physical assets and enter into an asset swap to achieve exposure to international equities; and
- a combination of the above.

The trade-off between the four approaches is determined by tracking error, transactions and administrative costs and liquidity requirements. *Exhibit 15.7* illustrates how this trade-off might work for a portfolio of approximately AU$300 million.

Exhibit 15.7

Cost-Benefit of Four Approaches to International Equity Indexing

	Physical Shares	SPI Futures	Asset Swaps*	Combination
Transactions Costs	1.03%	negligible	0.50%	0.10%
Administrative Costs	0.17%	0.06%	0.03%	0.05%
Liquidity	good	very good	very poor	good
Tracking Error	0.70%	1.80%	0.25%	0.50%

* Includes costs of maintaining the domestic component of the swap, but not the benefits of domestic dividend imputation credits.

Holding physical shares gives reasonable tracking performance with acceptable liquidity. The main impediment to holding physical assets is transactions and administrative costs. While most developed markets have fairly efficient exchanges and settlement systems, some still impose very high costs on foreign investors. International custodian charges are generally levied at a percentage of the amount under management as well as a fixed fee per transaction. For example, of the total annual administrative costs shown in *Exhibit 15.7*, all but about 0.05% are levied on a fee per transaction basis. This means that holding physical shares is much more expensive for a small portfolio than for a large one. Administrative and transaction costs for physical shares vary enormously from country to country. For example, the costs of holding physical shares in the United States are similar to those for domestic Australian holdings, while some smaller Asian and Scandinavian countries have costs which preclude holding any physical shares.

For a small to medium size portfolio, futures offer a cost efficient means of gaining international equities exposure. This cost efficiency is particularly important if the portfolio has frequent cash flows because the transactions and administrative costs thus incurred are very small. The problem with futures is that of tracking error. Tracking error has two sources. The first is due to the fact that not all countries in the MSCI have liquid share price index futures contracts. The second is because futures contracts do not match the MSCI index within the country.

Countries in the MSCI world benchmark index which have liquid share price index futures contracts cover about 93% of the benchmark by market capitalisation. Some of the countries which do not, such as Singapore and, until recently, Malaysia,[13] have in the past performed very differently to the index overall. Investors who lacked exposure to these markets would have underperformed the benchmark by almost 2% during 1993 and 1994.

Japan provides a good example of the other source of tracking error. This is because the Nikkei 225 contract, the most popular SPI futures contract for the Japanese equity market, is a price weighted index. This differs from the MSCI Japan index which is capitalisation weighted. This means that the smallest stock in the Nikkei 225 can move the index just as much as the largest, resulting in large performance differences between the Nikkei 225 and the MSCI Japan index (nearly 2.5% during 1993 and 1994). Japan does have futures contracts based on capitalisation weighted indices but, partly because of peculiar margining requirements, these have so far proved less popular than those settled to the Nikkei 225, so can introduce liquidity problems.

The use of asset swaps has increased considerably in the last few years. This is unsurprising given the obvious difficulties for the international investor of using physical shares and futures contracts. Asset swaps, if constructed efficiently, can overcome many of the cost and tracking performance problems normally encountered by the international indexer. They have the added charm of being potentially tax efficient too. Tax efficiency derives from the fact that the investor can hold domestic physical assets which may generate dividend imputation tax credits. The return on these assets are then swapped for the return on the desired basket of international assets.

Two important features of asset swaps are that they are not liquid and there exist significant scale economies. Asset swaps can present liquidity problems because they are all dealt as over-the-counter transactions, and so are difficult to terminate or reverse before their prearranged expiry date. The costs of early termination of a swap can be punitive. This means that a portfolio which holds a large proportion of its assets in swaps will be unable to meet a redemption without incurring very high costs. The economies of scale associated with asset swaps derive from the fact that they are always over-the-counter and usually customised to meet the demands of a particular investor. For this reason the set-up documentation and associated costs are normally high. These fees are the same in dollar terms regardless of the face value of the swap, so a large portfolio has a distinct cost advantage in percentage of total assets over a smaller one.

A combination of physical shares, futures and asset swaps can produce the best of all worlds, as can be seen from *Exhibit 15.7* above. *Exhibit 15.8* shows how this would look for a portfolio which requires 10% to be held in liquid assets.

13. Futures contracts started trading in Malaysia in late 1995.

Exhibit 15.8

Construction of International Equities Portfolio

	Benchmark	Physical	Asset Swap	Futures	Total
Europe G5*	23.20%	0.00%	0.00%	23.20%	23.20%
Rest of Europe	6.00%	0.00%	6.00%	0.00%	6.00%
North America	46.30%	41.67%	0.00%	4.63%	46.30%
Japan	18.60%	0.00%	16.74%	1.86%	18.60%
Hong Kong	2.00%	0.00%	0.00%	2.00%	2.00%
Rest of Asia	3.90%	0.00%	3.90%	0.00%	3.90%
	100.00%	41.67%	26.64%	31.69%	100.00%

* G5 = UK, Germany, France, Switzerland and Netherlands.

Europe G5 is invested in SPI futures because these give acceptable tracking performance and the costs of physical portfolios are relatively high in some of these countries. The rest of Europe uses an asset swap because the appropriate futures contracts are not generally available and the cost of holding physical assets is high. North America is invested 90% in physical assets because these can be held cheaply, with 10% in futures to meet liquidity requirements. Japan is held 90% in an asset swap to gain cost effective tracking performance, with 10% in SPI futures to meet liquidity requirements. Hong Kong, like Europe G5 is invested all in SPI futures, with the rest of Asia using an asset swap to gain cost effective tracking performance. The portfolio therefore meets the overall liquidity requirement of 10%, with 10% available in liquids for each of the major regions of Europe, North America and Asia. Because the portfolio is 26% invested through asset swaps, it has some capacity to earn domestic dividend imputation credits.

International equities—emerging markets: emerging markets present an entirely different range of challenges. The markets comprising the benchmark index do not have liquid derivatives markets, so the indexer has to decide between physical shares and asset swaps. Moreover, each country needs to be treated as a separate case. For example, many emerging countries discriminate between local and foreign investors, so that foreigners may in fact be precluded from purchasing certain stocks, or they may face discriminatory transaction costs or settlement procedures. Such conditions apply in one form or another in Korea, Philippines, Mexico and Venezuela. On the other hand, some countries offer parcels of their stocks, usually issued by intermediaries such as investment banks, which can be purchased in a unitised structure and may be traded on the stock exchanges of developed countries. While these instruments can simplify the administrative aspects of investing in emerging markets (and some developed markets), they usually incorporate many of the costs which face any foreign portfolio investor. This is because the intermediary, itself a foreign portfolio investor, must pass on the costs incurred in forming the underlying investment.

The composition of the MSCI World Emerging Markets Index is set out in *Exhibit 15.9*.

Exhibit 15.9

Composition of MSCI World Emerging Markets Index

Country	% of index	Mkt Cap USD Bn	GDP USD Bn	No of Stocks
Argentina	2.76%	27.3	268.8	23
Brazil	10.58%	104.7	584.6	61
Czech Republic	1.06%	10.5	40	20
Chile	3.58%	35.4	56.6	32
China	0.39%	3.9	744.9	26
Colombia	0.61%	6	70.3	9
Greece	1.08%	10.7	85.7	36
Hungary	0.31%	3.1	42.3	9
India	5.04%	49.9	327	67
Indonesia	4.42%	43.8	190.1	39
Israel	1.69%	16.7	87.9	51
Jordan	0.11%	1.1	6.4	15
Korea	9.10%	90.1	435.1	116
Malaysia	14.24%	141	80.4	76
Mexico	6.83%	67.6	304.6	42
Pakistan	0.45%	4.5	59.5	32
Peru	0.93%	9.2	55	14
Philippines	3.37%	33.4	72	35
Poland	0.40%	4	107.9	18
Portugal	1.78%	17.6	97	24
South Africa	9.82%	97.2	130.9	53
Sri Lanka	0.06%	0.6	12.6	10
Taiwan	14.93%	147.8	260.8	77
Thailand	4.76%	47.1	159.8	76
Turkey	1.18%	11.7	165.4	45
Venezuela	0.51%	5	65.5	13
Total	100.00%	989.9	4511.1	1019

Source: Morgan Stanley Capital International EMF and Emerging Markets Perspective, November 1996.

To the extent that the indexer invests in physical shares, a stratified sampling approach which tries to match country weights to the benchmark and industry groups within each country will be favoured.

The indexer will seek a combination of asset swaps, listed unitised vehicles and physical shares. The precise composition of the final portfolio will depend on the availability and price of each instrument for each region or country.

The aggregate market coverage of this portfolio is likely to be about 80%. This should allow a tracking error of between 1.5% and 2.0% pa, with a liquidity level of about 10%. Because transactions costs are high, rebalancing will be timed to coincide with cash flows.

8. FIXED INTEREST INDEX PORTFOLIOS

Fixed interest index portfolios require similar considerations as equity indexes. The indexer must first choose between a full replication or a sampling approach, then apply optimisation wherever possible. Mean-variance optimisation is less easily adapted to fixed interest portfolios, because time series data for individual asset returns are less readily available. Most fixed interest assets are traded outside normal exchanges, usually on screen-traded or "telephone" markets. Such trading systems may not include a mechanism for recording trades, so building the type of data base on which a portfolio optimiser works is simply not possible for many fixed interest markets. The Australian fixed interest market is one of these.

The indexer can, however, construct a stratified sample which does almost as good a job, provided care is taken to match duration as well as credit quality, and of course the interaction between the two.

The biggest difference between fixed interest and equity indexes is that fixed interest assets have a defined maturity date, while equity assets generally do not. This means that assets in fixed interest benchmarks can cease to exist from one day to the next.[14] They are usually replaced of course, but the problem remains that the composition of the benchmark index can change suddenly, although usually not without notice. The indexer is thus obliged to purchase the new asset regardless of the price it commands on the day.

Domestic fixed interest is benchmarked to the SBC Warburg Semi-Government All Maturities bond index. For this portfolio, the indexer has decided on a stratified sample. *Exhibit 15.10* shows each asset in the benchmark and the weight each is assigned in the portfolio. *Exhibit 15.11* summarises credit quality and duration of the 17 assets in the portfolio versus its benchmark.

The target tracking error is 0.10%. Liquidity of the portfolio is not a big issue as each security is sufficiently liquid to meet the 10% liquidity requirement.

14. This contributes to the problem of the lack of continuous return data.

Exhibit 15.10

Sample Index Portfolio for Domestic Fixed Interest

Description	Maturity	Coupon	Yield	Benchmark Weight	Portfolio Weight
Cash					1.46%
NSWTC	1/04/97	12.50%	7.10%	2.81%	0.00%
NSWTC	1/02/98	7.50%	7.18%	5.39%	7.35%
NSWTC	1/07/99	11.50%	7.31%	4.07%	5.99%
NSWTC	1/02/00	7.00%	7.36%	3.56%	0.00%
NSWTC	1/12/01	12.00%	7.53%	6.57%	9.68%
NSWTC	1/04/04	7.00%	7.75%	5.67%	8.99%
NSWTC	1/05/06	6.50%	7.94%	3.51%	0.00%
NSWTC	1/05/06	12.60%	7.94%	0.39%	0.00%
QTC	14/05/97	8.00%	6.85%	2.53%	0.00%
QTC	15/05/97	12.00%	6.85%	0.22%	0.00%
QTC	14/07/99	8.00%	7.05%	3.97%	6.15%
QTC	14/08/01	8.00%	7.24%	4.89%	6.61%
QTC	15/08/01	12.00%	7.24%	0.40%	0.00%
QTC	14/05/03	8.00%	7.40%	5.22%	7.07%
QTC	15/05/03	10.50%	7.40%	0.54%	0.00%
QTC	14/06/05	6.50%	7.60%	4.52%	5.22%
QTC	14/09/07	8.00%	7.81%	2.86%	0.00%
SAFA	15/10/96	12.50%	6.69%	1.86%	0.00%
SAFA	15/03/98	12.50%	6.82%	2.20%	4.16%
SAFA	15/10/00	12.50%	7.06%	1.88%	0.00%
SAFA	14/01/03	10.00%	7.27%	1.99%	2.53%
TASCORP	15/03/98	12.50%	6.82%	1.37%	0.00%
TASCORP	15/01/01	12.50%	7.09%	1.58%	3.68%
TASCORP	15/11/04	9.00%	7.44%	0.74%	0.00%
TCV	15/09/97	12.50%	6.78%	2.16%	0.00%
TCV	22/10/98	12.00%	6.88%	3.00%	7.72%
TCV	15/09/99	10.25%	6.96%	2.54%	0.00%
TCV	15/07/00	12.50%	7.04%	2.58%	5.36%
TCV	22/09/01	12.00%	7.15%	2.19%	0.00%
TCV	15/10/03	12.50%	7.34%	2.79%	0.00%
TCV	15/11/06	10.25%	7.63%	3.40%	5.39%
WATC	15/01/97	10.00%	6.82%	1.65%	0.00%
WATC	1/04/98	12.00%	6.93%	2.35%	3.51%
WATC	15/04/99	9.00%	7.03%	2.07%	3.76%
WATC	1/08/01	10.00%	7.24%	1.89%	0.00%
WATC	15/07/03	8.00%	7.42%	2.51%	5.37%
WATC	15/07/05	10.00%	7.61%	2.12%	0.00%
				100.00%	100.00%

Exhibit 15.11

Summary of Exposure and Duration

Description	Benchmark Weight	Portfolio Weight	Benchmark Duration	Portfolio Duration
NSWTC	31.97%	32.00%	1.1415	1.1416
QTC	25.16%	25.05%	1.1222	1.1220
SAFA	7.92%	6.69%	0.1917	0.1808
TASCORP	3.68%	3.68%	0.1174	0.1289
TCV	18.67%	18.47%	0.6701	0.6701
WATC	12.59%	12.64%	0.4226	0.4224
CASH	0.00%	1.46%	—	—
Total	100.00%	100.00%	3.6657	3.6657

This portfolio has been matched by credit quality, duration and duration within issuer. The stratified sampling process has taken no account of historical returns or their correlations; it assumes that each bond issue is fairly priced in the market and will remain close to fair price. Deviations from this rule will be the main contributor to tracking error for this portfolio. In practice these are likely to be small because each asset in the portfolio is liquid so mispricings should be quickly traded out.

International fixed interest is benchmarked to the Salomon Brothers World Government Bond Index. The indexer with access to a reliable source of historical data for the component securities in this benchmark will ideally choose an optimised sampling approach, with a target tracking error of about 0.30%. The benchmark index includes the 15 government bond markets of Australia, Austria, Belgium, Canada, Denmark, France, Ireland, Italy, Japan, the Netherlands, Spain, Sweden, Switzerland, the United Kingdom and the United States. In principle, the stratified sample approach will work just as well for this index portfolio as for the domestic fixed interest portfolio. In practice, the number of securities and the volume of data to be analysed favours a more automated version of this approach.

9. DOMESTIC LISTED PROPERTY SECURITIES

This sector is benchmarked to the Australian Stock Exchange Listed Property Accumulation Index.

It is generally accepted among indexers that direct property cannot be indexed because the benchmark consists of a small number of large assets, which are traded so infrequently that regular estimates of asset values are impossible. These obstacles have not deterred the occasional brave attempt to establish indexes of direct property holdings and even derivatives markets based on these. Success has been elusive.

So property indexes and index portfolios are nearly always confined to listed property markets. For the serious property indexer this is highly unsatisfactory because listed property securities do not behave like property markets. This could be because of the liquidity of the listed vehicles which is lacking in the underlying bricks and mortar, or because the prices of listed assets are more likely than direct property assets to incorporate expected future returns to the underlying assets. Or it could be due to other, unidentified factors. The returns to listed property securities are no more similar to those of other listed equities, so the asset class is a useful diversifying element in the portfolio.

From the point of view of the indexer, listed property markets can be treated in very much the same way as other equity index portfolios, except that their liquidity can be a little patchy. The indexer therefore will select a simple optimisation approach for this portfolio, choosing to invest in about three-quarters of the number of issues in the benchmark index, depending on the market's overall liquidity. Tracking error of 0.25% is feasible with negligible turnover.

10. ONGOING MANAGEMENT

It has been said that managing index portfolios is something like patrolling the Bay of Biscay in a Sunderland during World War II. Long periods of tedium are punctuated with bursts of manic activity which end as abruptly as they began. As it is unlikely that any individual has first hand experience of both occupations, the comparison must remain conjecture.

The indexer's aim is to maintain portfolio exposure close to benchmark while keeping costs to a minimum. If the portfolio is well constructed, changes in the composition of the benchmark which come about through normal price fluctuations will not cause problems, as the portfolio's composition will follow automatically.

Changes in the benchmark which are due to external factors may need some action. The guiding principle is that if the capital structure of the benchmark or one of its components changes, then the portfolio needs to follow suit. If there is no change in capital structure, then the indexer can take no action.

Examples of day-to-day changes to benchmarks are:

- Takeovers require no action so long as the portfolio holds both the offeror and the offeree companies. The indexer will usually wait until the takeover proceeds to compulsory aquisition or the bid fails.
- Stock splits require no action because there is no change to the company or the benchmark.
- Cash dividends are used to accumulate cash. This cash must be invested across the portfolio as soon as possible.
- Stock dividends are accepted in the form of physical shares, as the dividend represents an increase in the issued capital of the company.
- Share buy-backs are accepted because the issued capital of the company is contracting.

From time to time the benchmark changes because new stocks are added and others are deleted. For fixed interest benchmarks these changes are known and can be anticipated. For equities the indexer must maintain the appropriate information links.[15] The replicating indexer must buy or sell stock when the index change occurs regardless of prices and costs. The sampling indexer may make a judgement about when and how to buy and sell stock as required by benchmark changes, but will usually implement the required trades within a few weeks of the benchmark change.

Cash flows need to be dealt with on a case-by-case basis. Small cash flows are treated in a similar fashion to cash dividends; that is, they are accumulated as cash and equitised where possible using futures contracts until there is a large enough pool to warrant a purchase of physical stock. This stock purchase can be used to rebalance the portfolio if necessary, so avoiding unwanted transactions costs. If the cash flow is large then the indexer may buy physical assets outright, unless the price of the futures contracts offers an arbitrage opportunity (see the later section on enhancements). Normally the indexer will view any large stock purchase as an opportunity to fine tune the balance of the portfolio thus helping to keep transactions costs to a minimum.

11. MEASURING PERFORMANCE

Investors who place their funds in index portfolios expect unspectacular results. Their investments are expected to rise and fall in value more or less to the same extent as the benchmark or market to which they are indexed.

Performance measurement is therefore relevant only with respect to the benchmark index. For any given period, such as a month, quarter or a year, simple performance comparisons are useful but limited. The apparent performance of the portfolio will depend on the particular period being measured, with little or no indication of what occurred before or after, or in the intervening period. Consider the following series of results for a domestic equity portfolio in *Exhibit 15.12*.

15. Most index services provide periodical information documents about the composition of and changes to the benchmark. These services are usually offered on a subscription basis.

Exhibit 15.12

Monthly Performance of Domestic Equities Index Portfolio

Month	Benchmark	Portfolio	Difference	Month	Benchmark	Portfolio	Difference
31/05/94	1.15%	1.00%	−0.15%	30/06/95	0.48%	0.65%	0.17%
30/06/94	−4.03%	−3.95%	0.08%	31/07/95	4.91%	4.91%	0.00%
31/07/94	3.72%	3.80%	0.08%	31/08/95	0.94%	0.80%	−0.14%
31/08/94	3.04%	3.00%	−0.04%	30/09/95	0.74%	0.65%	−0.09%
30/09/94	−3.88%	−4.05%	−0.17%	31/10/95	−2.34%	−2.27%	0.06%
31/10/94	1.29%	1.36%	0.07%	30/11/95	4.43%	4.43%	0.00%
30/11/94	−7.23%	−7.06%	0.16%	31/12/95	2.53%	2.33%	−0.20%
31/12/94	1.65%	1.80%	0.15%	31/01/96	3.93%	3.85%	−0.08%
31/01/95	−4.25%	−4.16%	0.09%	29/02/96	0.39%	0.38%	−0.01%
28/02/95	5.07%	5.00%	−0.07%	31/03/96	−2.28%	−2.18%	0.10%
31/03/95	0.07%	0.02%	−0.05%	30/04/96	4.38%	4.20%	−0.18%
30/04/95	7.78%	7.98%	0.20%	31/05/96	−1.87%	−1.89%	−0.02%
31/05/95	−1.19%	−1.28%	−0.09%	30/06/96	−0.59%	−0.30%	0.29%

Measured to 30 June, 1996, the performance of this portfolio and its benchmark would look like *Exhibit 15.13*.

Exhibit 15.13

Performance Summary, Domestic Equities Index Portfolio to 30 June, 1996

Period	Benchmark	Portfolio	Difference
3 months	1.83%	1.92%	0.09%
6 months	3.82%	3.93%	0.11%
12 months	15.83%	15.53%	−0.31%
2 years	22.44%	22.73%	0.29%

The same performance measurements taken one month earlier would look like *Exhibit 15.14*.

Exhibit 15.14

Performance Summary, Domestic Equities Index Portfolio to 31 May, 1996

Period	*Benchmark*	*Portfolio*	*Difference*
3 months	0.10%	0.00%	−0.09%
6 months	7.08%	6.67%	−0.41%
12 months	17.08%	16.63%	−0.45%
2 years	18.21%	18.24%	0.03%

Neither set of results gives a good indication of the performance of the portfolio relative to the benchmark. In fact, it is not obvious that the two return summaries refer to the same portfolio. Obviously some form of continuous measurement is required to show how well the portfolio is tracking its benchmark. Because the CAPM tells us that asset returns fluctuate all the time but asset risk characteristics are more steady, some measure of the riskiness of the portfolio is required. Fortunately, there are several of these. Together they can indicate how much total risk is in the portfolio. Separately they tell us how much of that risk is due to the portfolio's exposure to its market, in other words its *beta*, and how much is due to other influences, collectively known as *tracking error*.

As we discovered earlier in the chapter, *beta* is a measure of the senstivity of the portfolio to the ups and downs of its market. To arrive at the observed beta measurement for a portfolio, a regression analysis is carried out. This will give an estimate of the historical beta of the portfolio as well as an estimate of how accurate the beta estimate is. This measurement, known as the R square, provides an aggregate measure of the variation between actual results and those predicted by the beta. An R square of 1.0 indicates that the beta calculated explains all of the movement of the portfolio against its benchmark, while an R square of zero says that the beta explains none of the portfolio's movement against its benchmark. In the example above, the beta of the portfolio for two years to 30 June, 1996 is 1.0012 and the R square is 0.9996. For the two years to 31 May, 1996 the beta is 1.0011 also with an R square of 0.9996. This means that, over the previous two year period, the portfolio is moving slightly more than the benchmark, so a benchmark move—up or down—of $100.00 would be associated with portfolio move—up or down—of $100.12. The R square means that this beta explains 99.96% of the portfolio's market driven performance.

In addition to the observed beta, the indexer is usually also interested in what the prospective or ex-ante beta is of the portfolio. This is a useful measure for new portfolios or where an old portfolio has been reweighted. The ex-ante beta of the portfolio is simply the weighted average of the betas of the component securities in the portfolio. (As for portfolio betas, stock betas are calculated by conducting a regression analysis between the stock returns and benchmark returns.) The ex-ante beta of our porfolio is estimated as 1.0009.

As mentioned previously, *tracking error* is a measure of how much of the portfolio's performance is due to risk that is not explained by general market movements

To obtain a measure of observed tracking error the differences in return for each period are calculated. The tracking error is the standard deviation of these differences, usually adjusted to give an annualised figure. The observed tracking error from *Exhibit 15.13* is 0.4404% to 30 June and 0.3958% to 31 May.

A tracking error of 0.4404% pa means that there is a 68% probability that the portfolio's performance will be within that range of the benchmark. In other words, if the benchmark index rises by 15% in a year, then there is a 68% chance that the portfolio will gain between 14.5596% and 15.4404% in that year.

The risk of the portfolio is the variance of its returns. Expected portfolio variance can be estimated from the return variance on the assets comprising the portfolio and the relationship, or covariance, between these assets and the benchmark. The ex-ante tracking error of the example portfolio is 0.45%.

The important thing to note here is that beta and tracking error do not change much over time so, unlike simple performance comparisons, they are not dependent on the period being measured.

Other indications of the performance characteristics of a portfolio include risk analysis and factor analysis. Both are the output of portfolio optimisers: factor models; and industry group models respectively. Although not as widely recognised as beta and tracking error, risk and factor analysis can add insight because they highlight the sources of specific risk, or tracking error, in portfolios. Factor model optimisers deliver factor analyses which allow the investment manager insight into how much portfolio risk is attributable to exposure to various factors. A typical factor analysis would look like *Exhibit 15.15*.

Exhibit 15.15

Factor Analysis, Domestic Equities Index Portfolio

Portfolio Specific Risk (Tracking Error) *Factor*	*0.42%* *Contribution to Tracking Error*
Variability in Markets	−0.02%
Success	−0.02%
Size	0.05%
Value	0.01%
Leverage	−0.01%
Growth	−0.02%
Trading Activity	0.02%
Interest Rate Sensitivity	0.00%
Total Factor Risk	0.01%
Residual Specific Risk	0.41%

The same portfolio analysed using an industry mean model is as follows in *Exhibit 15.16*.

Exhibit 15.16

Risk Analysis using Industry Mean Model, Domestic Equities Index Portfolio

Portfolio Specific Risk (Tracking Error)	0.42%
Sector	0.02%
Residual Specific Risk	0.40%

Both analyses indicate that the portfolio is reasonably well diversified because identifiable factor and sector risk is small. The remaining tracking error can probably be reduced only by increasing the number of stocks in the portfolio.

Performance measurement for indexed portfolios is in many respects more critical than for actively managed portfolios, because the indexed portfolio can generally tolerate less variation from benchmark performance.

12. ATTRIBUTION ANALYSIS

If everything goes according to plan the indexed portfolio's performance will be very close to that of the benchmark, as predicted by the optimiser's risk analysis. Things do not always go so smoothly however, so an attribution analysis may be called for to find out just where things went awry.

As with optimisers, attribution analyses come in a number of varieties, but they do hold some things in common. They try to break the portfolio down into manageable chunks and then see which of these chunks contributed to the unwanted performance variation. In the case of a domestic equities portfolio, industry groups are often used as the basis for attribution analyses; for international portfolios, countries can be used. In extreme cases the attribution analysis may be carried out at the level of individual securities to see which one caused the problem.

To illustrate how an attribution analysis might look, the performance over one month of the *TOP100* portfolio described in Section 5 above is analysed. Remember that this portfolio was compiled simply by selecting the 100 largest of the 300-odd stocks in the benchmark index. It is measurably overweight in diversified resources, media and banks because these sectors claim a disproportionate number of large stocks within the benchmark index.

During the period in question, the portfolio returned +0.21% while the benchmark returned −0.04%. This analysis seeks to identify not only which industry groups contributed to performance variation, but how much of that was due to the portfolio's being over- or underweight in each industry group, and how much can be explained by stock selection within each industry group.

The first of these, the industry mismatch effect can be calculated as follows:

$$IM = (w_{ip}i - w_{ib}) . (r_i - r_b) \qquad (4)$$

The second effect, the stock selection effect can be calculated for each industry group as follows:

$$SS = w_{sp}. (r_{sp} - r_{sb}) \qquad (5)$$

Where:

w_{ip} is the weight of industry *i* in the portfolio
w_{ib} is the weight of industry *i* in the benchmark
r_i is the return to industry *i*
r_b is the return to the benchmark
w_{sp} is the weight of stock *i* in the portfolio
r_{sp} is the return to stocks in industry *i* in the portfolio
r_{sb} is the return to industry *i* in the benchmark

The results of the analysis are set out in *Exhibit 15.17*.

Exhibit 15.17

Attribution Analysis for TOP100 Portfolio

	Industry Group Mismatch	Stock Selection Mismatch
Gold	0.06%	−0.12%
Other Metals	0.00%	−0.02%
Div Resources	0.04%	−0.01%
Energy	0.00%	0.11%
Infra & Utilities	0.00%	0.00%
Developers	0.00%	−0.04%
Building Materials	−0.01%	−0.01%
Alcohol & Tobacco	0.01%	0.03%
Food	0.00%	−0.02%
Chemicals	−0.01%	−0.02%
Engineering	0.00%	0.00%
Paper & Packaging	0.00%	−0.01%
Retail	0.00%	−0.04%
Tspt	−0.02%	−0.01%
Media	0.00%	0.01%
Banks	0.14%	0.17%
Insurance	0.00%	0.01%
Telecommunications	−0.01%	0.00%
Inv Services	−0.01%	0.00%
Property Trust	0.00%	0.01%
Misc Services	0.03%	−0.01%
Misc Industrials	0.01%	0.03%
Div Industrials	0.00%	−0.03%
Tourism	0.01%	−0.01%
Total	0.24%	0.02%

The first of these, the industry mismatch effect can be calculated as follows:

$$IM = (w_{ip}i - w_{ib}) . (r_i - r_b) \qquad (4)$$

The second effect, the stock selection effect can be calculated for each industry group as follows:

$$SS = w_{sp} . (r_{sp} - r_{sb}) \qquad (5)$$

Where:

w_{ip} is the weight of industry *i* in the portfolio

w_{ib} is the weight of industry *i* in the benchmark

r_i is the return to industry *i*

r_b is the return to the benchmark

w_{sp} is the weight of stock *i* in the portfolio

r_{sp} is the return to stocks in industry *i* in the portfolio

r_{sb} is the return to industry *i* in the benchmark

The results of the analysis are set out in *Exhibit 15.17.*

Exhibit 15.17

Attribution Analysis for TOP100 Portfolio

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Energy	0.00%	0.11%
Infra & Utilities	0.00%	0.00%
Developers	0.00%	−0.04%
Building Materials	−0.01%	−0.01%
Alcohol & Tobacco	0.01%	0.03%
Food	0.00%	−0.02%
Chemicals	−0.01%	−0.02%
Engineering	0.00%	0.00%
Paper & Packaging	0.00%	−0.01%
Retail	0.00%	−0.04%
Tspt	−0.02%	−0.01%
Media	0.00%	0.01%
Banks	0.14%	0.17%
Insurance	0.00%	0.01%
Telecommunications	−0.01%	0.00%
Inv Services	−0.01%	0.00%
Property Trust	0.00%	0.01%
Misc Services	0.03%	−0.01%
Misc Industrials	0.01%	0.03%
Div Industrials	0.00%	−0.03%
Tourism	0.01%	−0.01%
Total	0.24%	0.02%

The first of these, the industry mismatch effect can be calculated as follows:

$$IM = (w_{ip}i - w_{ib}) \cdot (r_i - r_b) \qquad (4)$$

The second effect, the stock selection effect can be calculated for each industry group as follows:

$$SS = w_{sp} \cdot (r_{sp} - r_{sb}) \qquad (5)$$

Where:

w_{ip} is the weight of industry i in the portfolio
w_{ib} is the weight of industry i in the benchmark
r_i is the return to industry i
r_b is the return to the benchmark
w_{sp} is the weight of stock i in the portfolio
r_{sp} is the return to stocks in industry i in the portfolio
r_{sb} is the return to industry i in the benchmark

The results of the analysis are set out in *Exhibit 15.17*.

Exhibit 15.17

Attribution Analysis for TOP100 Portfolio

	Industry Group Mismatch	Stock Selection Mismatch
Gold	0.06%	−0.12%
Other Metals	0.00%	−0.02%
Div Resources	0.04%	−0.01%
Energy	0.00%	0.11%
Infra & Utilities	0.00%	0.00%
Developers	0.00%	−0.04%
Building Materials	−0.01%	−0.01%
Alcohol & Tobacco	0.01%	0.03%
Food	0.00%	−0.02%
Chemicals	−0.01%	−0.02%
Engineering	0.00%	0.00%
Paper & Packaging	0.00%	−0.01%
Retail	0.00%	−0.04%
Tspt	−0.02%	−0.01%
Media	0.00%	0.01%
Banks	0.14%	0.17%
Insurance	0.00%	0.01%
Telecommunications	−0.01%	0.00%
Inv Services	−0.01%	0.00%
Property Trust	0.00%	0.01%
Misc Services	0.03%	−0.01%
Misc Industrials	0.01%	0.03%
Div Industrials	0.00%	−0.03%
Tourism	0.01%	−0.01%
Total	0.24%	0.02%

The portfolio benefited from being overweight banks because this sector performed better (+4.47%) than the benchmark (−0.04%) during the period in question. Within this sector the portfolio also benefited from holding large banks in preference to small banks. It also did well to be overweight in diversified resources, although stock selection in this sector contributed little to performance variation because the portfolio held three of the four stocks in the industry group. On the other hand, the portfolio benefited slightly from being underweight gold stocks, but did less well than it would have had it held more small gold stocks rather than only large ones. The portfolio's overweight position in media contributed little to performance variation because the performance of this sector during the period in question was not very different from that of the benchmark index.

It should be noted that this attribution analysis is unusually simple because the portfolio did not trade during the period. When transactions are taken into account an extra column is needed. This column is usually headed "market timing" or "trading activity" and tries to capture value added or subtracted from stock purchases and sales

13. INDEX ENHANCEMENTS

Index enhancements have grown in popularity over the last half-decade. There are a number of possible reasons for this. Probably the most important is the perception by many investors that markets harbour enough inefficiencies to allow risk-free extra market returns, and that indexed portfolios can be well placed to benefit from this "arbitrage" activity. Indexed portfolios bear a closer resemblance in structure to the benchmarks against which derivative instruments, such as share price indices, are settled and because indexing, as a "buy and hold" strategy, can tolerate substitution of individual or groups of stock holdings by futures and options for indeterminate periods of time. Investment managers favour enhancements because they attract higher fees than plain indexing.

Enhancements come in two varieties: risk free and risky. Some might say that a third category is at least as important as these two: risky enhancements masquerading as risk free.[16] If it is so, it is sure to be a short-lived activity[17] and will not be addressed in this chapter.

13.1 Risk free enhancements

An exhaustive treatment is not feasible because so many enhancements are possible, but the main sources of risk free enhancements are mispriced derivatives—such as share price index futures and options; listed stock options; dividend re-investment plans; and tax anomalies.

Mispriced derivatives enhancements characterise immature markets, and markets with particularly high transactions costs. The diversity of strategies

16. These can take many forms. Often they combine bought and sold positions in different securities, so that the portfolio's overall level of market risk is unchanged. This is not risk-free because the portfolio is exposed to increased specific risk due to the security mismatches introduced by the enhancement.
17. Because any additional returns will only compensate for the risk added.

which come under this heading is very rich, so a simple specimen will be used to represent the genus. The most straightforward is known as stock index arbitrage. This takes advantage of stock index futures which are trading at a price below "fair price".

Example: stock index arbitrage

Consider the following investment environment:

Date now	July 7
Physical share price index	2202
SPI futures	2215
Expiry date of futures	December 31
Dividend yield	3.2% pa
Interest rate	6.8% pa

The investor has two ways of investing in the stockmarket to the end of December: buy shares or buy futures and place the cash on deposit. The example below looks at what happens in each case.

It is important to note that the level at which the stockmarket closes on December 29 is irrelevant because the futures contract and the physical will be at exactly the same level at that date, however for the sake of illustration we will say that the market closes at 2210.

Strategy	Buy Shares	Buy Futures
Profit (loss) on shares	8	0
Profit (loss) on futures	0	(5)
Dividend income	34	0
Interest income	0	72
Profit (loss)	42	67
Percentage of initial investment	1.9%	3.0%
	42/2202 = 1.9%	67/2202 = 3.0%

In this case the investor holding cash and wishing to invest in the equity market is clearly better off using the futures contract. Another investor who already holds physical shares would sell these in favour of futures, so long as the transactions costs thus incurred are less than 1.1% (3.0 − 1.9) on the round trip. Most mature markets do not allow such easy profits: share price index futures contracts tend to trade in a range—determined by transactions costs—about their fair price. Less sophisticated markets can offer rich pickings in arbitrage activity, but this often comes with exotic sources of risk such as inscrutable trading rules and byzantine settlement systems.

Dividend re-investment plans (DRPs): many listed companies offer their shareholders the opportunity to receive dividends in the form of shares instead of cash. For the company concerned this activity has the advantage of either reducing the effective cash paid out in dividends or encouraging new equity investment in the company—whichever way one likes to look at it. Either way, new shares are issued in lieu of cash dividends. For the investor the attraction is that shares, which probably would have to be purchased anyway, are effectively bought with dividends forgone, saving transactions costs and often adding to return, because the DRP shares are normally issued at a discount to the prevailing market price of the share.

Tax anomalies: other enhancements seek to take advantage of anomalies between tax regimes. Usually this activity centres around the fact that different classes of investor are subject to different tax treatments, especially as they relate to dividends, imputation credits and withholding tax.

The most simple strategy is where the non-payer of local tax, such as a foreign investor, sells stock to a local taxpayer immediately before the ex-dividend date and repurchases the stock immediately after. Such transactions usually have some kind of repurchase agreement attached to protect both parties from unwanted swings in the share price. This may involve some sharing of the imputation benefit to give the non-local taxpayer incentive to carry out the transaction. Alternatively, the stock may be transferred as part of an asset swap or a stock lending arrangement. The principle of transferring the benefits of dividend imputation remains the same in each case but tactics differ in administrative and legal aspects. The only fly in the ointment is that, in some jurisdictions, uncertainties about interpretation of taxation law can seriously modify potential gains from this activity.

13.2 Risky enhancements

Most enhancements add risk as well as return. It is important, therefore, that the nature of the risk being added is understood and quantified. Enhancement strategies aiming to add risk controlled return are many and varied. Most seek to exploit a judgment about which securities or groups of securities will do better than others. Normally this judgment is the result of rigorous and sophisticated analysis. Most enhancement strategies therefore have predefined "rules" about when to buy and sell specified groups of assets, and a well articulated strategy for risk control while the strategy is working, and for damage control when it is not. It is these predefined rules and strategies which differentiate most index enhancements from conventional active portfolio management, but in some cases this distinction is one of degree rather than of definition.

Enhancements are generally designed to exploit perceived mispricing of some class of assets. They may also exploit mispriced derivatives such as stock index futures, as in the example above, or options on stocks. Options can present significant opportunities for enhancements because the investor can benefit not only from movements in the asset price but also from mispriced volatility. This is because the price of any option depends not only upon the price of the underlying asset but also, among other things, on the expected volatility of the underlying asset. If the volatility implied by the option price changes over time, the investor can benefit from correctly anticipating the change.

More specifically, if the volatility implied by the price of a particular option series is less than the actual volatility, then the investor can add value. This is done by purchasing the underpriced option—either call or put—and selling or buying the appropriate quantity of physical shares, or another option series which is more fairly priced (or vice versa if the option is overpriced). This position must then be managed closely as changes in the share price as well as the simple passage of time will introduce new, unintended risks to the position. Even when the position is meticulously managed, the investor risks losses if the original volatility estimate was inaccurate. Where the net option position is bought rather than sold, such potential losses can be limited to a quantifiable sum which is set at a level which is tolerable to the investor. Net sold positions, on the other hand, can result in potentially large losses as the maximum loss on a sold option position is usually unlimited.

Index enhancement strategies which employ physical assets must always be carefully managed because, as with sold options positions, the potential risks to the portfolio are very large. Such enhancements are often referred to as portfolio tilts. This implies that the portfolio deviates only slightly—is "tilted" away from true index proportions. The direction of the tilt can be determined by factor considerations, macro-economic variables or active security analysis.

Factor tilts suggest a tilt to a factor which affects the performance of some stocks relative to others. An example of such a tilt would be toward growth stocks rather than value stocks. The portfolio would therefore be expected to outperform in market environments favouring emerging companies, such as those in growth industries like high technology or telecommunications. Conversely, the portfolio will underperform if sectors which represent "value"—which might include countercyclical stocks such as discount retailing and food manufacture—outperform the growth sectors.

Tilts which are driven by macro-economic factors seek to exploit superior economic analysis. This usually results in an opinion about the equilibrium level of some macro-economic variable, such as interest rates. Thus an investor anticipating a change in the level of interest rates might favour stocks which are shown to be sensitive to interest rate changes such as banks and financial services or very capital-intensive industries. An expected fall in oil prices might favour transportation stocks, and so on.

Tilts deriving from active security analysis simply exploit the perceived mispricing by the market of individual securities. This type of tilt is the one which most resembles conventional active management.

Cash enhancements can also add value to an index portfolio if the portfolio happens to hold enough cash. Fortunately, there are enough occasions for this activity (usually where cash is held in the portfolio to provide collateral for derivatives positions) to allow significant value to be added. Cash enhancements add value by adding two types of risk to the portfolio: yield curve risk; and credit risk.

Yield curve risk is added by purchasing interest bearing assets with more than one day to maturity. Under normal circumstances such assets earn higher rates of interest than cash (overnight securities) simply to compensate the investor for the additional risk associated with tying up assets for a longer period. The risk to the investor is that interest rates will rise in the meantime, so increasing the cost of capital and/or the opportunty cost of not having funds to invest at a higher yield.

Credit risk is the risk that the issuer of the interest bearing securities is unable to fulfil its obligations, or will undergo a credit "rerating", which will cause the market value of its issued debt to fall. This is due to the fact that the interest earned on such securities reflects the market estimate of the credit risk for any issuer of debt, so that the higher the interest earned the higher the credit risk.

14. CUSTOMISING INDEX PORTFOLIOS

Index portfolios are particularly suitable to customising because there is no expected outperformance, or alpha to be compromised. The investor may choose to customise either the benchmark, the portfolio or both.

Standard benchmarks are much more popular than customised ones because they are easily measured and widely available. On the other hand, customised benchmarks often provide a more meaningful basis of performance evaluation, particularly if the portfolio specifications are somewhat unusual.

The most popular forms of customisation relate to the level of risk or costs that the portfolio can sustain. In such circumstances the indexer will usually determine the number of securities in the portfolio to give the required results. Or the investor may have specific liquidity requirements and this will determine the portfolio's rebalancing schedule and the use of derivatives.

If the portfolio is held in a jurisdiction where differential tax treatment applies to different classes of investor, the portfolio may be customised to meet specified after-tax objectives.

The investor may have reason to impose embargoes on particular stocks and so demand a customised benchmark. For example, an ethical fund may wish to avoid the arms or tobacco industries. Customised portfolios and customised benchmarks are particularly useful where asset classes overlap, causing a potential double exposure to some asset class. This is often the case where the widely-used benchmark for equities includes both large cap and small cap, but where the investor needs to treat these as separate asset classes. In this instance two customised benchmarks would be constructed. The first would comprise the broad-based benchmark, excluding some collective measure of the small cap stocks. The second would be the broad-based benchmark excluding large cap stocks.

15. TRAPS FOR YOUNG PLAYERS

Constructing and managing index portfolios can look deceptively simple, and some index portfolios are indeed very easy to look after. This perception is reinforced by the continuing downward pressure on management fees.

The biggest pitfall is the expectation that one approach fits all. Often this means that the indexer goes for full replication without considering the implications for the costs of running the portfolio. Probably the next biggest is that of using the optimiser or other stock selection software as a "black box". Experienced indexers will choose their optimiser according to the market to be indexed and will also exercise some care about how data is sourced and applied.

Once the portfolio is up and running the indexer will need to take care to manage liquidity and to avoid unwanted portfolio turnover. Often these are two sides of the same coin because they incorporate the use of derivatives, such as share price index futures.

Corporate actions can cause problems by landing the portfolio with securities which are simply not required for diversification and which merely add to transactions and administration costs. Pre-emptive action is occasionally required to avoid this happening. Similarly, large changes to the benchmark holdings, often resulting from privatised utilities or other publicly owned entities, can present the indexer with quite a challenge. These problems need to be addressed on a case-by-case basis, and require a good understanding of the structure of the underlying market and any relevant derivatives markets.

Where the index portfolio holds derivatives, either by choice or by default, then the indexer must take care that any options holdings do not result, through delta expansion or contraction, in the portfolio being either geared or underexposed. While this problem can also occur in an actively managed portfolio, it is usually more serious in an index portfolio, which usually has less spare liquidity and generally much finer tolerances of performance variation.

16. SUMMARY

Indexation is the answer to the problem of security selection for investors who wish to avoid the costs or the risks of seeking and managing high alpha portfolios. Sometimes investors choose not to chase high alphas because they believe that they represent a wild goose chase.

More often indexation is selected for much more prosaic reasons. This may be that the investor has a very large portfolio and so employs the indexed core-active satellite approach. This effectively avoids both the very high costs of market impact that can result from a large active portfolio, as well as the situation where dividing the fund into many small active mandates results in a very high cost "closet index".

Other candidates for indexation include small or very cost sensitive investors who simply cannot afford active management, and international managers who want to concentrate their efforts on managing beta or market risks without the distracting effects of specific portfolio risks in obscure parts of the world. Still other index portfolios are run as part of a larger strategy deriving from some position in derivatives such as share price index futures or asset swaps, or to meet the requirements of customisation.

Indexation of portfolios will continue to grow as long as pressure on investment management fees continues, and while large funds continue to dominate markets relative to small- and medium-sized investors. Increased international investment will contribute to further growth in indexation, both through indexation of portfolio holdings and indirectly through the increased use of asset swaps as a tax effective means of managing international investments.

The concept of indexing is simple, but eliminating an acceptable amount of a portfolio's specific risk without the costs defeating the very purpose of indexing can be tricky. The best approach is nearly always specific to the situation, depending, among other things, on the market in question and the particular objectives of the investor. The indexer will aim for a balance of minimised cost and specific risk or tracking error, choosing either to fully replicate the target market or to aim for benchmark-like returns using some sample of securities. This sample will be selected either as a cross-section of the market or by using a mean-variance optimiser, or some combination of the two.

The indexer will typically adhere to a rebalancing schedule determined by fixed decision rules, taking into account the likely life span and cash flow requirements of the portfolio.

Performance of index portfolios is best described by evaluating the risk of the portfolio rather than by comparing returns over any given period. This is because the risk characteristics of assets can be thought of as being stable over time while returns are not.

Some index portfolios are subject to return enhancements, which may or may not add alpha risk. Normally enhanced index portfolios seek to exploit mispriced derivative instruments, but some apply wisdom distilled from specialist quantitative or macroeconomic analysis. Still others add value by exploiting anomalies in tax regimes.

Indexation is particularly useful in the context of customised portfolios because it can meet customising requirements without jeopardising the portfolios' other objectives.

The simplicity of indexing is sometimes deceptive and the temptation to apply textbook or black box solutions should be avoided. The skill of the indexer is to achieve the theoretical benefits of indexing while avoiding the practical traps.

Part 5

Risk Management

Chapter 16

Value at Risk Models[1]

by Satyajit Das and John Martin

1. INTRODUCTION

Market risk has emerged as a central issue in financial risk management in recent years. Value at risk (VAR) techniques have also emerged as a central mechanism for quantification and communication of risk on multiple levels ranging from individual traders, trading units to entire businesses, both within financial institutions, and albeit to a lesser degree, in non-financial institutions.

In this chapter, the concept of VAR and its application in market risk management is examined. The structure of this chapter is as follows: the evolution of risk management is considered first. The basic principles of market risk and its measurement is examined later. The VAR technique is then outlined, including consideration of the various types of VAR, the application of VAR to different types of instruments and the interpretation of the VAR statistics in risk measurement as well as consideration of the assumptions underlying the derivation of VAR measures. The application of VAR based risk management techniques to financial institutions and non financial institutions is then considered. The subject of risk adjusted performance measurement is also considered.

2. EVOLUTION OF RISK MANAGEMENT

2.1 Financial risk

All financial intermediation entails the assumption, management and pricing of risk. Risk, in this broader sense, encompasses credit risk, market risk, liquidity risk and operational risks. These risk classes are essential to the process of credit, maturity and market (primarily, interest rate) risk intermediation central to all financial activity.

Credit risk refers to the risk of loss arising from the *default* of the counterparty ie the failure to honour and meet its legal obligation. Market risk refers to the risk of loss sustained as a result of changes in the values of market prices or factors used to value financial instruments. Liquidity risk

1. The writing of the chapter was undertaken separately though the authors collaborated on the ideas behind the underlying concepts. Satyajit Das is responsible for the main text of the chapter while John Martin is responsible for Appendices A and B as well as Exhibits 16.8, 16.9, 16.10, 16.11, 16.12, 16.13, and 16.15. The authors would like to thank Suellen Schmidt for her assistance in the preparation of some of the exhibits. The authors would like to thank J P Morgan (Debra Robertson) for consent for the inclusion of the material on measuring the risk of managing option position using analytic VAR and simulation methods in Exhibit 16.17.

refers to the risk of loss arising from either inability to make payments or the inability to re-finance obligations as and when they mature or the inability to re-finance at anticipated rates. Operational risk refers to the risk of loss from a broad range of risks including: operational (processing failure); technology (systems failure); legal (non unenforceability of contracts); and regulatory (breach of regulatory requirements).

Underlying the identified risk factors is the broader issue of management failure or strategic risk wherein the value of the business (in present value terms) is affected by individual risk factors. Examples of this type of failure include the reputation risk where entry into transactions which were not suitable leads to litigation which has the effect of damaging the client perception of the entity.

The important point to make is that trading and market risk management is a part of *the overall risk profile of an entity.*

2.2 Key factors in the evolution of financial institution risk management practice

Most if not all these risks have been present in the process of financial intermediation all along. Several factors have contributed to the increased focus on risk management:

- the deregulation of financial markets;
- the increasing role of securities and derivative products in financial intermediation;
- the increase in the risk profile of organisations, with increased emphasis on activities which require the assumption of risk, deliberately;
- the volatility of markets and its impact on financial institutions;
- the pressure from capital market investors for returns related to the relative riskiness of their investments; and
- the regulatory requirements for a framework for the management of risk.

It is salutary to remember that financial intermediation industries throughout the world have only been de-regulated relatively recently. European markets have only been de-regulated in the last 10-15 years and emerging markets are still subject to varying degrees of regulation.

The presence of regulation impacts on risk in at least two ways: first, it may regulate risk taking itself; and, secondly, the risk taking activity may, to varying degrees be underwritten by the financial system at large *reducing* the requirement for risk management in individual organisations. Elimination of regulation emphasises both risk taking and risk management by individual organisations.

The increased role of securities and derivative products in financial markets affects trading risk management in a number of ways. The increased use of securities in financial markets has resulted in bank financing being gradually supplanted by bonds and other tradeable obligations. This trend has included the securitisation of many hitherto illiquid assets such as mortgages and various types of receivables, extending from credit card, motor vehicle etc loans and lease obligations. The impact of securitisation has included an emphasis on evaluating the risk of these obligations through periodic revaluation of the position, by marking positions to current market values. This trend has replaced, to a large degree traditional methodologies of classical accrual accounting.

The increased role of derivatives in transferring risk and synthesising exposure to asset prices has promoted a focus on risk management for different reasons. The dynamic value characteristics of these instruments, including the potential for leverage, has dictated the need to both frequently and regularly mark to market positions but also to monitor and manage the *potential* value changes to these products as a result of changes in asset prices or other variables.

The increase in trading activities of firms and the resulting volatility of income and the overall increase in risk of the firm's activities has been much commented upon. The reasons underlying this increase in trading include:

- the fall in agency revenues as markets deregulate;
- the increasing pressure from clients for execution based on the financial institution trading as principal rather than agent; and
- the perceived competitive advantage enjoyed by financial institutions in trading, including infrastructure, information, transaction costs etc.

The assumption of risk is often presented as a relatively new phenomenon. In reality, the *nature* of risk taking has changed. Traditional risk taking was confined to balance sheet oriented interest rate risk created by deliberate maturity mismatches. Supplementary risk taking may have been present in areas like currency or securities trading.

Modern risk taking is more diverse in nature. It encompasses traditional forms of risk taking and increased trading in other asset classes. This increased trading focus is driven by both client demands in terms of market-making and proprietary or own account trading, in search of return. This change reflects in no small part the increasing diversity of the activities undertaken by financial institutions which encompass activities in all asset classes (debt, currency, equity, and commodities) and in businesses ranging from balance sheet driven activities (such as lending), off balance sheet activities (such as underwriting and securities distribution and risk management instruments), to pure fee based activities (such as investment management, custody services, cash management services etc).

Underlying the focus on risk is the impact of volatility on values of financial instruments as a result of the change in financial market asset values. While there is little evidence that *overall* levels of volatility have increased it is evident that volatility levels have often become compressed into short time periods when volatility levels rise very substantially. The impact of volatility combined with the increased risk profile of financial institutions dictates a more comprehensive focus on risk management.

The pressure to manage risk is also evident from investor demand for:

1. Understanding of risks taken by financial institutions in a unified and transparent framework; and
2. Measures of return relative to risk for *all activities* of financial institutions.

The external pressure, from increasingly activist shareholders, has its counterpoint in increased internal pressure to measure risk adjusted returns to enable more accurate evaluation of performance and also allocation of capital as between business units.

The final driver of risk management is the regulatory process itself. The transition from a regulated to a deregulated environment requires adjustments in regulatory tools. Traditional tools, such as balance sheet constraints, interest rate controls or specific regulations, regarding participation in specific activities, have gradually been supplanted by capital based controls, requiring minimum capitalisation levels commensurate with risk.

The 1988 Capital Adequacy Accord represents the first concrete step in that process requiring capital resources consistent with the degree of *credit risk*. The credit risk capital framework inevitably creates an adverse incentive in that credit risk taking is discouraged while market risk taking (against which capital does not need to be held) is encouraged. In reality, capital based market risk controls were inevitable from the time the capital based credit risk guidelines were mooted in the mid-1980s.

Underlying the evolution towards risk management is also the process of innovation in capital markets where the product life cycles have shortened and the pace of introduction of new products, reflecting the earnings potential available for the first mover, has increased. Against this background, a more flexible risk management framework was inevitable.

2.3 Risk management versus asset-liability management

It is incorrect to assume that institutions had *no* risk management previously. In fact, most financial institutions had in place frameworks for risk within an overall asset liability management (ALM) framework. The ALM model is now less and less relevant and increasingly replaced by an overall risk management framework. The differences between the two models are considerable:

- ALM is balance sheet oriented while risk management is risk oriented;
- ALM is not easily adapted to off-balance sheet transactions, particularly derivatives;
- ALM risk measures which are interest rate based (gaps and duration) are not necessarily easy to translate across other asset classes;
- ALM provides little mechanism for arriving at an overall risk level for a firm and is not easily adapted to a risk adjusted return framework; and
- ALM, unlike risk management, is difficult to link to either other risks or to return on the firm's equity capital.

Risk management in effect reshapes ALM into the context of modern financial intermediation. Several aspects of this current approach are well worth noting:

- Risk in the modern context of mean variance finance is subject to considerable statistical assumptions which users need to appreciate in utilising the information; and

- While developed largely in the context of trading, particularly derivatives trading operations, it is universally applicable to all aspects of financial intermediation.

2.4 Financial risk management in non-financial institutions

The concepts outlined, while developed primarily for financial institutions active in trading, are equally applicable with some modest adjustments to non-financial institutions. However, the nature of financial risk in non-financial institutions is significantly different.

These differences relate to:

- the financial versus non-financial nature of the underlying assets and liabilities; and
- the source of the underlying risk and the ability to manage this risk.

These differences dictate the nature of risk management processes and approaches for the different types of institutions. An understanding of the identified differences is central to the proper analysis of risk and the concomitant application of derivative instruments to manage that risk.

For financial institutions, the fact that *both* assets and liabilities are financial in nature is central to the task of risk management. This allows risk positions to be matched and the net position to be hedged. The process of hedging is facilitated to a substantial degree by the essential financial nature of these exposures which are correlated to some tradeable financial market variable or index. This allows the risk to be managed synthetically by taking an offsetting position in the physical market or through derivatives.

For non-financial institutions, either the assets in the case of industrial corporations or the liabilities in the case of investment portfolio managers are non-financial in nature. For industrial corporations, the underlying assets include real assets, such as property, plant and equipment, intangible assets, such as goodwill (surplus on acquisition), intellectual property and brand names, as well as financial assets in the form of equity or other investments. For investors, the liabilities may be linked to mortality (life insurance policies), casualty events (fire, earthquake or other physical events) or indexed to inflation or cost-of-living changes.

The most significant aspects of these non-financial assets and liabilities are:

1. they are effectively financed by financial assets or liabilities which have fixed servicing and repayment profiles;
2. the return profile and value dynamics of the non financial assets or liabilities are not driven by the same value drivers that determine the value of the financial assets and liabilities that finance them;
3. the use of mark-to-market methodology to measure risk, at least in a current and static sense, may not be appropriate or even capable of implementation for non financial institution. This will be the case for the non-financial asset and liabilities meaning that marking to market the offsetting financial liabilities and assets may provide an incomplete and potentially misleading profile of risk; and

4. the capacity to utilise value-at-risk type approaches to model risk on a prospective basis will be constrained for non-financial institutions.

This inherent mismatch reduces the scope to create *net* exposures and, more importantly, the capacity to hedge these exposures with traditional hedging instruments.

The source of risk is also fundamentally different. For a financial institution, risk is to a large degree a matter of choice. The risk assumed is as a result of *creating* the mismatch through a course of dealing—choosing *not* to offset an exposure as a result of a transaction entered into. An example of this would be a foreign exchange transaction entered into with a client which the financial institution chooses not to offset with an equal but opposite transaction. Given the imperfection of certain markets, such as low liquidity and high transaction costs, it may be difficult to *exactly* match the transaction but a surrogate transaction to, at least, lower the exposure, through the principle of a correlation based hedge as noted above, is generally available.

In contrast, for a non-financial corporation, the risk is *inherent* to a substantial degree arising from substantially non-financial sources, such as the business carried out or the business strategy being pursued. For example, for an oil producer, the risk to the commodity price is one which is inherent from the business activities.

The major implications of this source of risk include:

- financial risk for financial institutions is transitory while for non-financial institutions it is permanent;
- the perpetual nature of the risk for non-financial institutions dictates that it is inherently difficult to hedge reflecting the finite maturity characteristic of risk management products;
- the use of capital to manage the risk of loss from financial risk which is mandatory within financial institutions is not applicable in the same manner to non-financial institutions; and
- the measurement of hedge performance is likely to be very different as between the two types of organisations.

The central dominating feature of corporate financial management is the concept of hedging or management of financial exposures to limit the risk to the *cash flow* of the company. This reflects, in part, the central role of cash flow, in modern finance, in determining the value of assets. The acceptance of shareholder value approaches (such as SVA, EVA, MVA, CFROI, et cetera) as the basis of shareholder value, performance measurement metric and, ultimately, strategy formulation is a component of this trend. The need to mange risk in terms of cash flow also highlights its central role in managing the risk of bankruptcy or financial distress of the company.

The acceptance of cash flow as the basis for risk management brings with it a series of distinct problems. For example, it does not necessarily solve the problem of the non-financial assets. However, these approaches do allow capture of a more complete risk profile of non-financial institution's exposures and facilitates their management than traditional risk management methodologies.

3. MARKET RISK AND ITS MANAGEMENT

3.1 Sources of market risk

The concept of risk in finance may be defined as one of uncertainty of values. Specifically, market risk can be defined as the risk to an entity of losses resulting from adverse changes in financial asset prices, including changes in interest rates, currency rates, equity prices, and commodity prices.

Market risk exists in many forms. In the case of a financial institution, this risk arises from financial transactions entered into for the purpose of facilitating a client's requirements (eg to buy or sell assets), financial assets held in inventory or transactions entered into to deliberately expose the institution to the asset price movements in the expectation of being able to profit from anticipated favourable movements. These transactions may be transactions involving securities or off-balance transactions such as derivatives.

In the case of an investor, the exposure arises from the risk of loss in value of investments (direct or indirectly through derivative products) from adverse movements in the value of asset

In the case of non-financial institutions, such as industrial corporations, the risk relates to the financial attributes of business transactions. This may include:

- exposure to interest rates changes on its portfolio of liabilities and liquid investments.
- risk from currency value changes on its revenue or expenses (where these are denominated in a currency other than its functional currency) or on the value of its foreign currency denominated assets and liabilities. The revenue and expenses items as well as assets and liabilities affected are both financial and operating in nature.
- exposure to commodity price changes, either direct or indirect, which impact upon revenue or on its cost of production or operation.
- risk from changes in equity values on equity investments and fund raising.

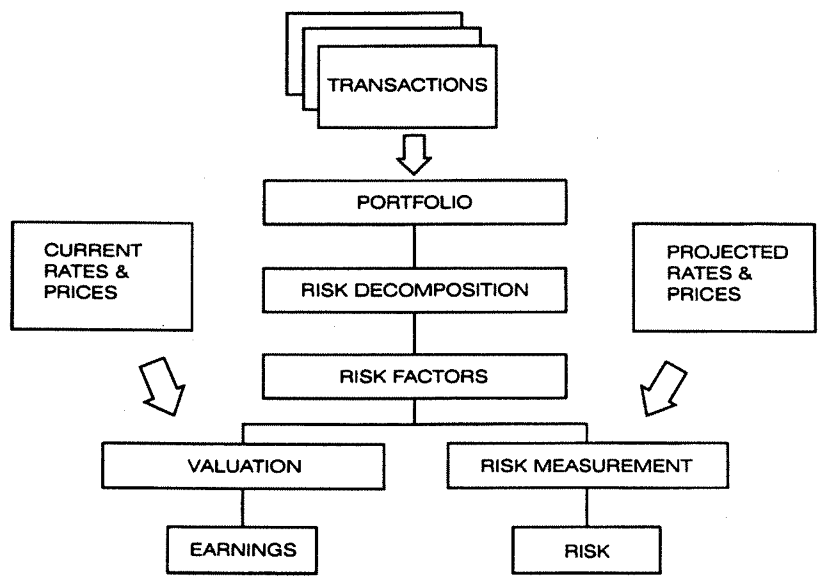
In each of the above cases, the exposure to market risk necessitates measurement of the quantum of the exposure. This measure is an essential precursor to the management of the risk, either through elimination of the exposure through hedging action, including the use of derivatives, or maintaining capital against the risk to protect the entity from the risk of unacceptable loss.

3.2 Market risk model

The process of measuring market risk entails a series of distinct and separate steps. A market risk measurement model is set out in *Exhibit 16.1*.

Exhibit 16.1

Market Risk Measurement Model



As set out in the model, the market risk arises from the transactions or positions entered into which are first aggregated to create a portfolio of transactions. The objective of the aggregation process is to combine similar transactions and net positions to the maximum degree possible to arrive at a net or marginal risk position.

The portfolio is then decomposed into the underlying risk factors. The process of risk decomposition entails the breakdown of each instrument, irrespective of its structure or format (on balance sheet or derivative), into its pure risk components (debt/interest rate risk at selected yield curve points, currency risk, commodity price risk and equity risk). The process of risk decomposition as discussed in detail below, is central to the aggregation and consolidation of risk across products and across asset classes. This process of consolidation is essential to capturing the benefit of diversification of risk taking activities within the entity where the risks are non-correlated.

The decomposed portfolio is then processed in two separate ways:

1. *Valuation*—whereby the portfolio is revalued using current prices and rates for the relevant risk factors to estimate the earnings of the portfolio.

2. *Risk Measurement*—whereby projected prices and rates are used to estimate the risk of the portfolio.

The valuation of the portfolio entails marking the portfolio to market using current prices and rates. The mark to market process establishes the value of the portfolio on a liquidation basis. The value established is the value of the portfolio were it to be liquidated at the relevant moment of time. The mark to market value provides valuable information on the success or failure of the transactions entered into, the liquidation value of the portfolio, and the earnings of the portfolio (represented by the change in value of the portfolio from its previous valuation). This information provides the basis for decisions on future action in respect of the portfolio, including transactions to be liquidated or new transactions to be entered into. Where the transactions are designed to hedge certain risks, the mark to market process provides a measure of the performance of the hedge.

The central element of the valuation process within a mark to market framework is its emphasis on *current market prices*. This contrasts with traditional accounting measures of value which emphases accruals. The mark to market valuation approach is consistent with portfolios of liquid and tradeable instruments which allows dynamic management and presents continuous opportunities for value optimisation. The accruals methodology assumes a static framework of management and is consistent with maintenance of these portfolios to maturity.

Risk measurement is concerned with *projecting* futures prices and rates and using these projections to estimate the risk of loss of the portfolio. This can entail, as described more fully below, a wide variety of techniques designed to estimate possible *adverse* changes in market rates and prices and the measurement of the impact of these changes on the value of the portfolio. In essence, the measure of risk entails the following:

- Estimating the magnitude of any potential adverse changes in individual risk factors (the volatility of market prices) to allow the quantum of any loss to be estimated.
- The inter-relationship of changes as between market risk factors (the correlations as between changes in risk factors) to allow incorporation of any benefits of diversification of risks within the portfolio.

In addition, the degree of confidence in the estimate of the adverse change (the statistical confidence level) and the time period needed to reasonably efficiently liquidate the portfolio (the holding period) must also be nominated to allow the risk to be estimated.

VAR is a generic term which covers a group of similar techniques designed to specifically measure market risk within this framework.

3.3 Applications of market risk measurement techniques

The market risk management approach described, based on VAR techniques, allows the estimation of market risk inherent in an entity or activity. This facilitates a number of functions, including:

- management information and oversight;
- establishment of trading limits and control of trading operations;

- performance evaluation;
- asset and resource allocation, including hedging decisions; and
- regulatory reporting and risk oversight.

Information regarding the risk of activities provides the basis of management of operations exposed to market risk.

Integral to this process of management is the establishment of appropriate trading limits and controls of trading operations. This would entail the setting and monitoring of compliance with limits based on capital or earnings/cash flow at risk in trading activities from adverse market price movements. These limits can be designed for business, individual units as well as traders to allow appropriate risk oversight.

The ability to establish and monitor limits allows the linking of returns from market risk activities to the *capital risked* in the course of such activities. This allows the development of *risk adjusted* performance measurement systems enabling a proper dissection of contribution to overall earnings.

The information available also provides the basis for asset and resource allocation to, at a macro level, activities that provide higher risk adjusted returns allowing diversion of resources away from less attractive opportunities or businesses. At a micro level, the information allows the evaluation of individual transactions allowing trading strategies to be evaluated in terms of the expected return relative to risk allowing maximisation of risk return of trading opportunities. Similarly, it allows the development of hedging strategies based on risk reward trade-offs.

This type of information is integral to the process of regulatory reporting and risk oversight of market risk generating activities being implemented by central banks and other regulators of financial intermediaries. Central to these developments is the impending introduction of capital adequacy requirements in relation to market risk developed by the Bank of International Settlements and being implemented by central banks throughout the world. This follows the European Union's Capital Adequacy Directive (EEC 93/6) which also requires financial institutions to hold capital against market risks assumed in the course of their activities. Parallel developments are evident in relation to investment institutions and corporations where increased disclosure about the impact of market price fluctuations is now increasingly being required.

4. VALUE AT RISK TECHNIQUE

4.1 VAR concept

VAR can be defined as:

> "The expected loss on a position from an adverse movement in identified market risk parameter(s) with a specified probability over a nominated period of time."

It is calculated as:

VAR = Current Value of the Position *times* Sensitivity of the Position to a change in the relevant Risk Factor *times* Potential Change in the Risk Factor

The calculation of each component and the overall VAR requires the specification of the numeraire currency. In the case where the positions/instruments are denominated in a foreign currency, the foreign currency equivalents positions, sensitivities and VAR figure will need to be restated in the nominated base currency.

The current value of the position represents the mark-to-market value of the position revalued at current market rates and prices. In effect, the current value of the position specifies the size and direction (long or short) of the risk exposure.

The sensitivity of the position is the equivalent of the present value of 1 basis point (PVBP) or the dollar value of 1 basis point (DVO1) of the position. For example, the sensitivity of fixed interest instruments is the PVBP based on movements of the relevant interest rate. The sensitivity of other assets, currencies, equities, and commodities, will be related to the movements in the underlying price with a US$0.01 movement in the asset price will result in an equivalent change in the value of the position. The sensitivity measure is designed to provide the responsiveness of the position to changes in market prices of the relevant risk factors.

The potential change in the risk factor is central to the calculation of VAR. It is designed to capture potential changes in the relevant risk factors from current levels which will drive changes in the value of the position determining the risk of holding the position. It is, in effect, an estimate of the *volatility* of the relevant risk factor.

The critical elements in selecting the potential change in risk factor are:

- deriving the volatility estimate;
- nominating the level of confidence required; and
- selecting the risk horizon.

Volatility in financial markets is usually calculated as the standard deviation (σ) of the percentage changes in the relevant asset price over a specified period. The volatility for calculation of VAR is usually specified as the standard deviation (σ) of the percentage change in the risk factor over the relevant risk horizon.

As discussed in detail below, the volatility estimate can be derived in a number of ways. These include utilising historical volatilities from a selected time series of data on the risk factor or using implied volatilities from traded instruments.

The concept of the confidence level is designed to allow an estimate to be made regarding the probability that the change in the risk factor will *not exceed*, based on the assumptions made, a nominated level. The use of confidence levels is related to the definition of risk as the variability of the possible changes in the risk factor around an expected change. This assists in allowing the calculation of the level of change in the risk factor and as a consequence the value of the positions that is expected to occur no often than a certain amount of the time.

The selection of the risk horizon relates to the potential liquidation period of the position. Where a trading position is held, the elimination of the risk of holding the position is achieved by liquidating the holding or assuming an equal and opposite position (which in effect is equivalent to the first option). In measuring the risk of any position it is necessary to assume how quickly or slowly the risk can be eliminated as the relative risk of the position will be a function of this time period (the longer the potential holding period, the greater the risk).

The choice of the confidence level and holding period is usually the following:

- *Confidence level*—1.65 σ (equivalent to 95% confidence) or 2.33 σ (equivalent to 99% confidence).[2]
- *Time horizon*—1 day or more.

While the choice, in theory, resides with individual institutions, the following guidelines exist:

- The G-30 Report recommends the use of 95% confidence levels and a 1 day holding period.
- The BIS Market Risk Guidelines recommends the use of 99% confidence levels and a 10 day holding period.

4.2 Application of VAR to estimating market risk—single asset

Exhibit 16.2 sets out the formal mathematical definition of VAR. *Exhibit 16.3* sets out an example of calculating the VAR of a market risk position involving a single asset. The VAR is calculated for 95% and 99% confidence levels and for holding periods of 1, 10 and 30 days.

Exhibit 16.2

VAR Calculation—Single Asset—Definition

$$VAR = V_n . dV/dp . \sigma_{day} . CF$$

Where

V_n = market value of position n

dV/dp = sensitivity to price changes per $ market value

σ_{day} = volatility or standard deviation of daily price changes

CF = number of standard deviations consist with selected confidence levels (that is, 1.65 for 95% and 2.33 for 99%)

2. The confidence level utilised is statistically the cumulative normal distribution (equivalent to the area under the curve for a normal or Gaussian statistical distribution. In estimating risk, a one tailed distribution is usually used. This reflects that for risk purposes the focus is on potential risk of *losses* only. In these circumstances, 95% (99%) confidence levels are equivalent to 1.65 s (2.33 s).

Exhibit 16.3

VAR Calculation—Single Asset

Position

Assume an organisation has a foreign exchange position whereby it has purchased US$76,550,000 against a sale of A$100,000,000 for spot value. The objective is to quantify the risk of this position over a 1 day (overnight), 10 day and 30 day holding period.

Assumption

The transaction is opened at the current spot of A$1.00 = US$0.7655. The daily volatility (calculated as the standard deviation of the continuously compounded historic price changes) is estimated as 0.4364%.

VAR Calculation

The VAR of the position is calculated as follows:

Asset

Type	US$ Currency
Market Value	$ 76,555,000
Term	Spot

Daily volatility = 0.4364%

Value at Risk Calculation

Holding Period	Volatility	95% Confidence	99% Confidence
1 day	0.4364%	$ 551,196	$ 778,356
10 day	1.3799%	$1,743,035	$2,461,376
30 day	2.3901%	$3,019,025	$4,263,229

The VAR for a 1 day holding period is also often referred to as daily earning at risk (DEAR). The term DEAR has been popularised by J P Morgan.

The calculation of VAR for longer holding period entails the scaling up of the VAR estimate using the following relationship:

$$VAR_n = VAR_1 \times \sqrt{n}$$

Where

VAR_1 = VAR for a 1 day holding period

VAR_n = VAR for a n day holding period

n = holding period in days

This relationship assumes:

- the daily changes are random and follow a random walk; and
- the daily changes are not cumulative.

4.3 Application of VAR to estimating market risk—multiple assets

Extending the application of VAR to a portfolio consisting of multiple assets entails a similar logic to that utilised for a single asset. The only difference relates to the summation of the risks within the portfolio.

The simple addition of the risks is not appropriate as this assumes that the adverse events are likely to occur simultaneously. This is equivalent to assuming a correlation of 1.00 between the changes in the risk factors. In reality, the changes in the risk factors will rarely be perfectly correlated. In the case of similar risk factors, the direction of change may be the same but the magnitude will be less than one. For example, zero rates for different maturities in the same currency are likely to be positively but imperfectly correlated, implying a correlation of between 0 and 1. The changes in risk factors across asset classes may be *negatively* correlated, that is the change in one risk factor may be accompanied by an opposite change in the second risk factor. For example, an increase in interest rates may accompany a decrease in equity prices.

This requires the incorporation of the correlation between the changes in the risk factors in the calculation of VAR. *Exhibit 16.4* sets out the formal mathematical relationships.

Exhibit 16.4

VAR Calculation—Multiple Assets—Definition

Two asset portfolio

The calculation of the market value of the position and the sensitivity to changes in market price are identical to that for a single asset calculation.

The volatility estimate is given by:

$$\sqrt{[w_1\sigma_1{}^2 + w_2\sigma_2{}^2 + 2\,(w_1 w_2 \sigma_1 . \sigma_2 . \rho_{12})]}$$

Where

w_1 = portfolio weighting for asset 1

w_2 = portfolio weighting for asset 2

σ_1 = the volatility of asset 1

σ_2 = the volatility of asset 2

ρ_{12} = the correlation between price changes between asset 1 and asset 2

Multi asset portfolio

For a portfolio of n assets, the volatility estimate is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j P_{ij}}$$

Where

σ_p = the volatility of the portfolio of asset consisting of n assets

$w_i;w_j$ = the weighting for asset i and j in the portfolio

$\sigma_i;\sigma_j$ = the volatility of the changes in the price of asset i and j in the portfolio

ρ_{ij} = the correlation between the changes in the price of asset i to asset j

Exhibit 16.5 sets out an example of calculating the VAR of a market risk position involving two assets. The VAR is calculated for 99% confidence levels and for holding periods of 10 days.

4.3 Application of VAR to estimating market risk—multiple assets

Extending the application of VAR to a portfolio consisting of multiple assets entails a similar logic to that utilised for a single asset. The only difference relates to the summation of the risks within the portfolio.

The simple addition of the risks is not appropriate as this assumes that the adverse events are likely to occur simultaneously. This is equivalent to assuming a correlation of 1.00 between the changes in the risk factors. In reality, the changes in the risk factors will rarely be perfectly correlated. In the case of similar risk factors, the direction of change may be the same but the magnitude will be less than one. For example, zero rates for different maturities in the same currency are likely to be positively but imperfectly correlated, implying a correlation of between 0 and 1. The changes in risk factors across asset classes may be *negatively* correlated, that is the change in one risk factor may be accompanied by an opposite change in the second risk factor. For example, an increase in interest rates may accompany a decrease in equity prices.

This requires the incorporation of the correlation between the changes in the risk factors in the calculation of VAR. *Exhibit 16.4* sets out the formal mathematical relationships.

Exhibit 16.4

VAR Calculation—Multiple Assets—Definition

Two asset portfolio

The calculation of the market value of the position and the sensitivity to changes in market price are identical to that for a single asset calculation.

The volatility estimate is given by:

$$\sqrt{[w_1\sigma_1{}^2 + w_2\sigma_2{}^2 + 2\,(w_1 w_2 \sigma_1 . \sigma_2 . \rho_{12})]}$$

Where

w_1 = portfolio weighting for asset 1

w_2 = portfolio weighting for asset 2

σ_1 = the volatility of asset 1

σ_2 = the volatility of asset 2

ρ_{12} = the correlation between price changes between asset 1 and asset 2

Multi asset portfolio

For a portfolio of n assets, the volatility estimate is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j P_{ij}}$$

Where

σ_p = the volatility of the portfolio of asset consisting of n assets

$w_i;w_j$ = the weighting for asset i and j in the portfolio

$\sigma_i;\sigma_j$ = the volatility of the changes in the price of asset i and j in the portfolio

ρ_{ij} = the correlation between the changes in the price of asset i to asset j

Exhibit 16.5 sets out an example of calculating the VAR of a market risk position involving two assets. The VAR is calculated for 99% confidence levels and for holding periods of 10 days.

Exhibit 16.5

VAR Calculation—Multiple Assets—Example 1

Position

Assume an organisation (base currency A$) has a US$ asset position (a zero coupon 10 year bond) which it has purchased US$76,550,000. The position gives rise to two risks:

1. interest rate risk exposure to 10 year zero coupon interest rates; and
2. movements in the US$/A$ exchange rate.

The objective is to quantify the risk of this position over 10 day holding period.

Assumption

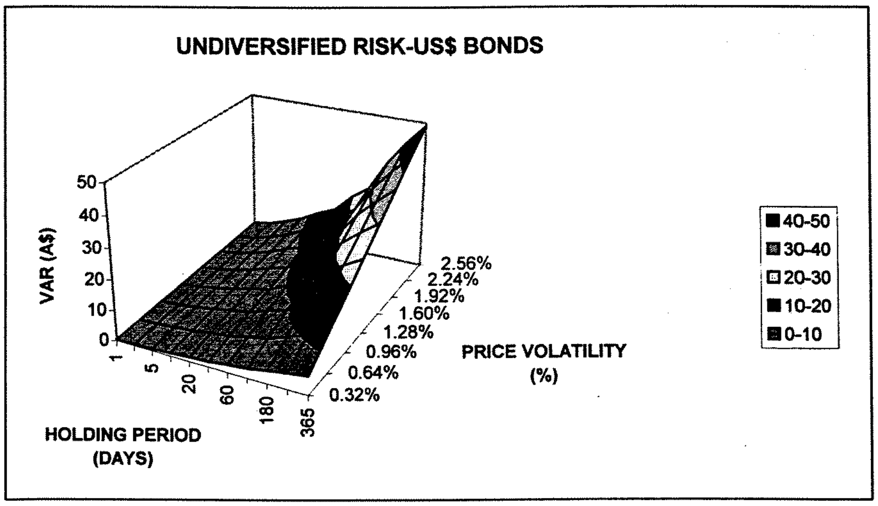
The transaction is opened at a price where security is valued at 100% of original purchase price and the current spot of A$1.00 = US$0.7655. The daily volatility (calculated as the standard deviation of the continuously compounded historic price changes) is estimated as 0.55% for the 10 year interest rate and 0.2630% for the US$/A$ currency rate. The correlation between the two variables is estimated at -0.15.

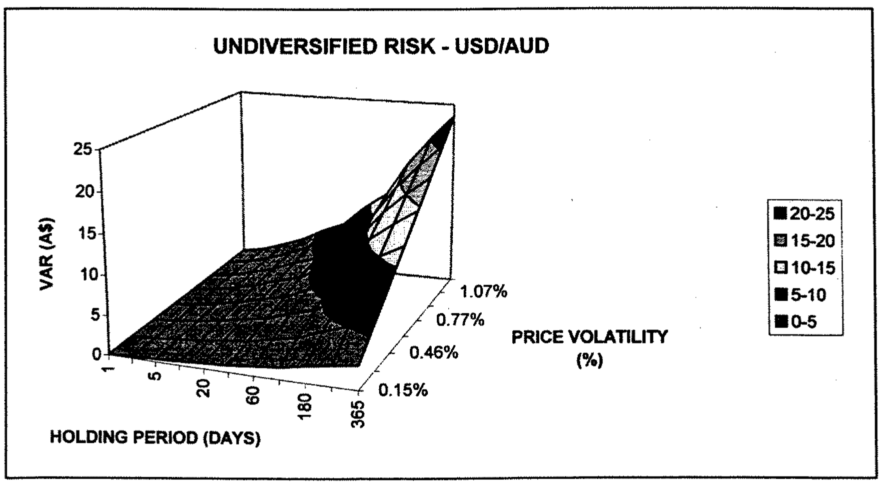
VAR calculation

The VAR of the position is calculated as follows:

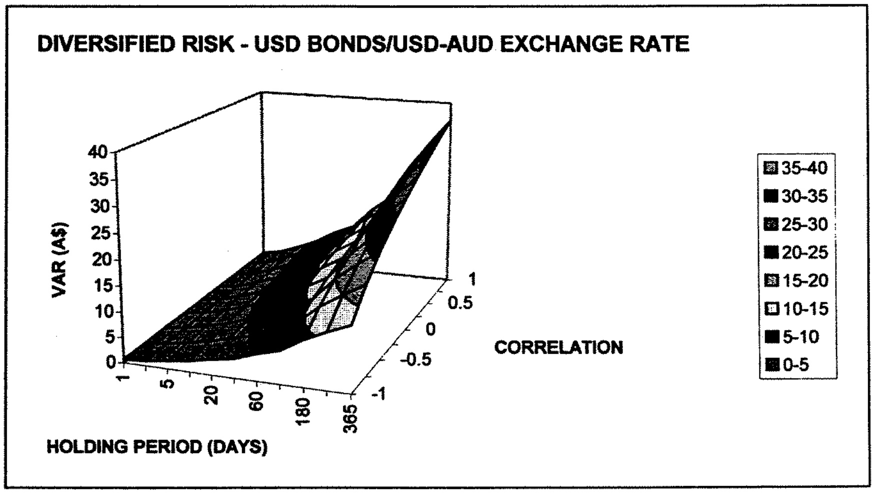
ASSET (FACE VALUE) (USD)	
POSITION AMOUNT (USD)	$76,550,000
CURRENT PRICE (USD)	$76,550,000
PRICE VOLATILITY (PER DAY)	0.5500%
PRICE VOLATILITY (OVER HOLDING PERIOD)	4.0525%
VALUE AT RISK (USD)	$3,102,157
VALUE AT RISK (AUD)	$4,052,459
USD/AUD EXCHANGE RATE	
POSITION AMOUNT (USD)	$76,550,000
CURRENT EXCHANGE RATE	$0.7655
CURRENT PRICE (AUD)	$100,000,000
PRICE VOLATILITY (PER DAY)	0.2630%
PRICE VOLATILITY (OVER HOLDING PERIOD)	$1.9380%
VALUE AT RISK (A$)	$1,938,035
VALUE AT RISK—UNDIVERSIFIED RISK	$5,990,494
VALUE AT RISK—DIVERSIFIED RISK	
PRICE VOLATILITY (OVER HOLDING PERIOD)	$4.2216%
VALUE AT RISK (A$)	$4,221,642
VALUE AT RISK REDUCTION (A$)	$1,768,852

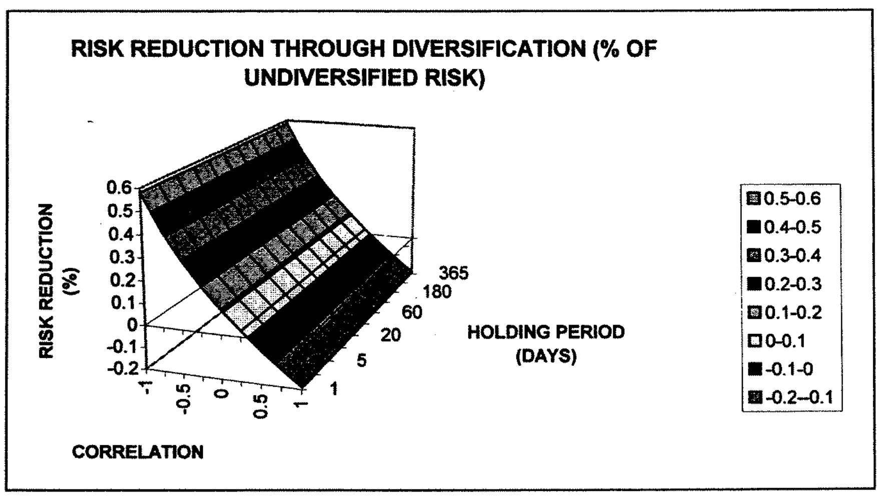
The sensitivity of the VAR of the individual interest rate and currency positions to changes in volatility and holding period are summarised in the following graphs:


UNDIVERSIFIED RISK-US$ BONDS
VAR (A$)
50
40
30
20
10
0
1
5
20
60
180
365
HOLDING PERIOD (DAYS)
2.56%
2.24%
1.92%
1.60%
1.28%
0.96%
0.64%
0.32%
PRICE VOLATILITY (%)
40-50
30-40
20-30
10-20
0-10


UNDIVERSIFIED RISK - USD/AUD
VAR (A$)
25
20
15
10
5
0
1
5
20
60
180
365
HOLDING PERIOD (DAYS)
1.07%
0.77%
0.46%
0.15%
PRICE VOLATILITY (%)
20-25
15-20
10-15
5-10
0-5

The following graph shows the sensitivity of the *diversified* VAR (incorporating correlations) of the total position to changes in correlations between the variables and the overall reduction in VAR achieved:





The process of aggregation and consolidation of risk is complex where large portfolios are involved. The process will generally require multiple levels of consolidation. This will be undertaken firstly within each asset class or sub-class with subsequent consolidation across asset classes. *Exhibit 16.6* sets out an example of this process.

Exhibit 16.6

VAR Calculation—Multiple Assets—Example 2

Position

Assume an organisation has a long position in Swiss Franc (CHF) bonds which are offset (partially hedged) by a short position in Deustchemark (DEM) bonds. The positions are predicated on the fact that CHF and DEM interest rates are closely correlated. The VAR of the position is required to incorporate the multiple level consolidations that are necessitated by the positions in two interest rates and two currency pairs.

VAR calculation

The VAR of the position is calculated to incorporate the various correlations at a number of separate levels:

1. correlations between yield curve points at different maturities;
2. correlations between interest rates in the different currencies;
3. correlations between the two currency pairing (against the US$); and
4. correlations between the interest rates and currencies.

The diagram set out below sets out the step by step consolidation process to measuring the risk of the combined positions:

AGGREGATING VAR USING CORRELATION TREE

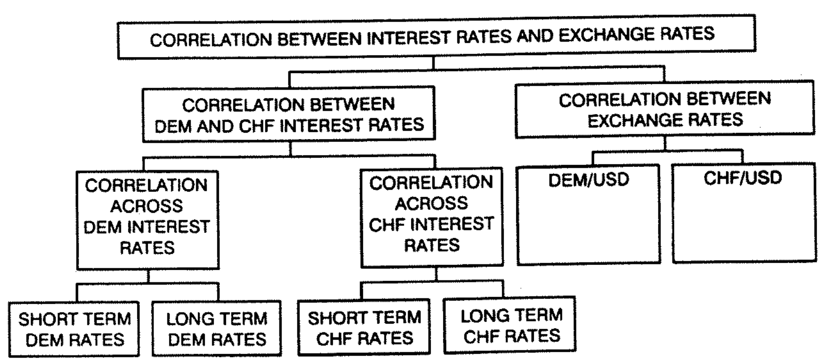


Exhibit 16.6—*continued*

The impact of the correlations in reducing the overall VAR of the combined positions is summarised in the table set out below:

RISK FACTOR	VAR	CORREL-ATION	1ST LEVEL CONSOLID-ATION	CORREL-ATION	2ND LEVEL CONSOLID-ATION	CORREL-ATION	3RD LEVEL CONSOLID-ATION
ST DEM RATES	10000	0.69	7483				
LT DEM RATES	−5000			0.61	10499		
ST CHF RATES	−15000	0.50	−13229				
LT CHF RATES	10000					0.19	15232
DEM/ USD	20000	0.90	9220		9220		
CHF/ USD	−15000						
TOTAL	75000		29932	19718	15232		

Source: This example is drawn from Chris Matten, "The Capital Allocation Challenge For The Banks" SBC Prospects 4-5/1995 at 4-5.

The calculation of VAR for portfolios of asset requires estimates of the correlations between changes in the prices of assets. The VAR calculation may be sensitive to these correlation estimates.

5. IMPLEMENTING VAR

5.1 Overview

The basic VAR concept is relatively simple. Implementation of VAR type approaches, in practice, require a number of distinct and separate steps:

- The decomposition of portfolios of transactions into the underlying risk factors.
- The incorporation of adjustments to the framework to accommodate non-linear price risk (primarily, option transactions).
- Estimation of the volatilities and correlation required.

5.2 Risk decomposition in application of VAR

5.21 Concept

The process of risk decomposition entails the process by which individual transactions and products are reduced into cash flows and the relevant factors to enable the measure of market risk.

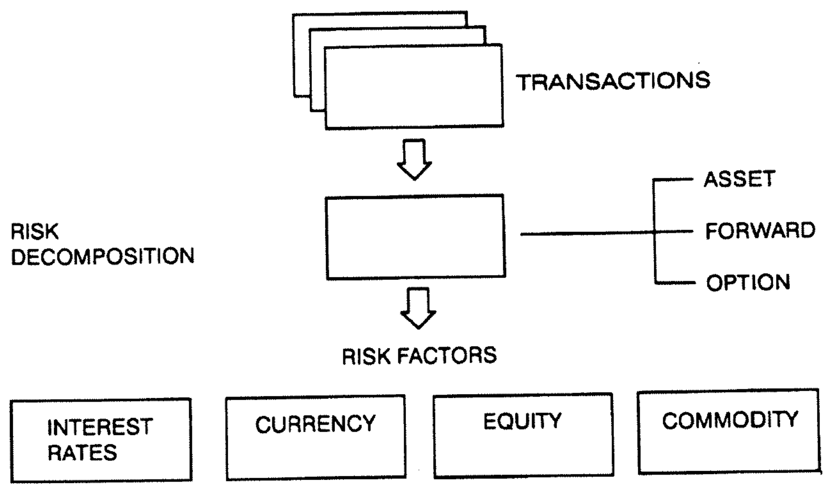
Exhibit 16.7 sets out the basic model of risk decomposition. The process consists of a number of distinct and separate steps:

1. Similar products and transactions are aggregated to establish net positions in each.

2. The cash flows of each transaction are identified and the products or transactions are then decomposed into the following components:
   (a) the asset class (debt/ interest rates, currency, equity or commodities);[3] and
   (b) the instrument itself—asset, forward, option.
3. The transactions are then restated in terms of the relevant risk factors.

Exhibit 16.7

Risk Decomposition Model



The process of decomposing the position in the asset class into the various instruments (asset or derivative) is designed to facilitate a second order reallocation of the cash flows and risks. For example, a forward on an asset can be decomposed into an asset risk and an interest rate risk reflecting the exposure to changes in the cost of the forward through the carry cost. This process is referred to variously as risk decomposition, risk mapping (J P Morgan in RiskMetrics™), cash flow mapping, and cash flow shredding.

The process of risk decomposition is predicated on a combination of the following factors:

- data requirements;
- flexibility; and
- computational efficiency.

The estimation of market risks using VAR would require either tracking the volatilities and correlations between the price changes on individual

3. It is increasingly feasible to add credit as a separate asset class, particularly with the advent of credit derivatives; see Satyajit Das (ed), *Credit Derivatives* (John Wiley & Sons, London, 1997).

instruments or a series of defined risk factors. The former approach would require information on a very large number of individual instruments. In contrast, the process of decomposing all risk positions into a selected number of risk factors would assist in reducing the amount of information required. For example, thousands of fixed interest instruments would be reduced a series of cash flows allocated to a number of maturity points.

This process also adds to the flexibility of the process. Where estimation of market risk is instrument specific, new products would require individual price volatilities and correlations to enable the market risk to be measured. This may in practice be very difficult particularly where the instruments are not liquid and the establishment of the volatilities and correlations are more difficult. The process of risk decomposition because it allows restatement of any new product in terms of established risk factors greatly increases the flexibility of this framework in accommodating non-standard transactions and innovations.

The computational efficiency of risk decomposition is also superior to that of competing approaches. The number of risk factors dictates the size and number of correlations required. Based on the relationship that the number of correlation is equal to [n (n-1)]/2 (where n is equal to the number of risk factors), the rapid growth in the number of correlations can be seen from the following Table:

Number of Risk Factors	Number of Correlations	Increase (times) in Risk Factors	Increase (times) in Correlations
20	190		
50	1,225	2.5	6.4
100	4,950	5.0	26.1
200	19,900	10.0	104.7
500	124,750	25.0	656.6

The use of a limited number of risk factors contains the number of correlations and facilitates efficient calculation of market risk even for very large portfolios.

Despite the fact that the process of risk decomposition may, in certain areas, be artificial and arbitrary, the very real benefits that can be achieved in terms of reducing data requirements, increasing flexibility, and enhancing efficiency makes this approach extremely desirable.

5.2.2 *Minimum risk factors*

The process of risk decomposition is intrinsically linked to the number of risk factors specified. The selection of risk factors is inevitably a compromise between the accurate and full capture of the market risk of the portfolio and the computational efficiency considerations identified.

In practice, the following minimum risk factors have been nominated by the BIS in its Market Risk Guidelines:

- *Fixed Interest*—to be represented by cash flows allocated to a number of maturity points (a minimum of six) sufficient to capture exposure to changes in zero and forward rates as well as changes in yield curve shape. The credit risk of fixed income securities is captured through the use of a minimum of two yield curves, namely, a risk free government curve and a credit adjusted yield curve (usually, the swap curve). The credit spread curve is required to be specified such that any specific credit risk is captured in the risk measures.

- *Currency*—represented by the relevant currency cash flows and the individual currency pairings.
- *Equity*—represented by the cash flows in the relevant currency of the position. At a minimum, the risk factor should be related to market wide movements in the relevant equity market, based on the equity market index, with individual stock positions being captured by utilising the beta (relative risk measure) of the stock. Depending on the nature of the trading operation and the risks assumed, more extensive risk factors—ranging from sectoral indexes to risk factors for individual stocks—may be required. The risk factors should encompass the stock specific risks such as the exposure to changes in dividend payments and changes in the stock's beta.
- *Commodities*—represented by cash flows in the currency of trading (usually US$) in the commodity. Risk factors should encompass both the directional price risk and other aspects of the commodity risk including basis risks and risks of changes in convenience yields.

The above risk factors are well adapted to measuring the risk of position in the assets or forward contracts. As described in greater detail below, the risk of option contracts incorporates additional risks which are not completely and effectively captured by the above risk factors.

The specific methodologies of decomposing risk factors by asset class is outlined in the following sections. In each case, the process of decomposing and restating the risk of asset and linear derivatives (forward transactions) is considered. Option transactions are not dealt with in relation to individual asset classes being covered separately in the section on application of VAR to option generally.

5.2.3 Risk decomposition—fixed interest

Fixed income securities are segmented into two fundamental categories: government securities and securities which entail specific credit risk (non government bonds). Within each category, the cash flows of the securities are stated in terms of:

- amount; and
- the timing, that is the date when they are due.

The basic approach requires each individual cash flow to be treated as a separate zero coupon bond which can then be valued using the relevant zero coupon rate (the government or risk free zero rate for government securities and the zero coupon swap rates for all other instruments).

If the bond has any embedded optionality (for example, callable or putable bonds), then the option element is isolated and treated consistently with the treatment of other options.

The major issue with the risk decomposition of fixed interest instruments is the allocation of cash flows to specific maturity points or vertices. This problem arises because of the limited number of maturity points usually used. For example, RiskMetrics™ uses some 18 maturity vertices in each currency ranging from overnight to 30 years.

The allocation process is usually designed to satisfy certain conditions including:

- Ensuring that the present value of the cash flows after allocation is the same as the market value of the instrument.
- The market risk of the cash flows (as measured by, for example, historical volatility) is equivalent to the market risk of the original cash flows.
- The allocated cash flows have the same direction (inflow or outflow) as the original cash flows.

The algorithms for allocating cash flows to individual maturity points are potentially very complex.

Exhibit 16.8 sets out an example of the process of allocating the cash flows of a fixed interest bond to a set of maturity points.

Exhibit 16.8

Risk Decomposition—Fixed Rate Bond

The attached table shows the process of allocating the cash flows of a fixed rate bond into a limited set of maturity vertices. The bond in question is specified in the section entitled Bond Details and is decomposed into a fixed number of vertices (11 in total). The algorithm utilised is that suggested by J P Morgan in its RiskMetrics product[4] which satisfies the three conditions noted. The decomposed cash flows are then utilised to calculate the VAR of the individual cash flows as allocated to individual maturity vertices which are then amalgamated utilising the correlation between the maturity points to calculate the fully diversified VAR of the bond.

Bond Details

Principal	1,000,000
Settlement	1-Jan-97
Maturity	1-Dec-06
Coupon %pa	8
Coupon Frequency	2

Undiversified VAR	6,731
Diversified VAR	6,652
Diversification Benefit	78

	1 Day	1 Week	1 Mo.	3 Mo.	6 Mo.	12 Mo.	2 Yr	3 Yr	4 Yr	5 Yr	7 Yr	9 Yr	10 Yr
Price Volatility	0.000	0.001	0.001722	0.005033	0.01603	0.057673	0.167344	0.268379	0.372298	0.464388	0.645418	0.812764	0.885251
Correlation Matrix													
1 Day	1	0.992	0.178	0.089	0.125	0.03	0.105	0.074	0.051	0.032	0.012	0.007	0.013
1 Week	0.992	1	0.284	0.15	0.178	0.066	0.127	0.095	0.068	0.049	0.03	0.025	0.03
1 Mo.	0.178	0.284	1	0.59	0.509	0.244	0.213	0.185	0.149	0.133	0.139	0.136	0.135
3 Mo.	0.089	0.15	0.59	1	0.744	0.528	0.391	0.384	0.361	0.348	0.322	0.309	0.312
6 Mo.	0.125	0.178	0.509	0.744	1	0.664	0.597	0.59	0.578	0.57	0.556	0.542	0.542
12 Mo.	0.03	0.066	0.244	0.528	0.664	1	0.853	0.863	0.854	0.845	0.806	0.777	0.772
2 Yr	0.105	0.127	0.213	0.391	0.597	0.853	1	0.99	0.969	0.948	0.889	0.847	0.842
3 Yr	0.074	0.095	0.185	0.384	0.59	0.863	0.99	1	0.991	0.976	0.928	0.89	0.885
4 Yr	0.051	0.068	0.149	0.361	0.578	0.854	0.969	0.991	1	0.996	0.961	0.931	0.927
5 Yr	0.032	0.049	0.133	0.348	0.57	0.845	0.948	0.976	0.996	1	0.978	0.955	0.951
7 Yr	0.012	0.03	0.139	0.322	0.556	0.806	0.889	0.928	0.961	0.978	1	0.994	0.989
9 Yr	0.007	0.025	0.136	0.309	0.542	0.777	0.847	0.89	0.931	0.955	0.994	1	0.998
10 Yr	0.013	0.03	0.135	0.312	0.542	0.772	0.842	0.885	0.927	0.951	0.989	0.998	1

4. See J P Morgan/Reuters, *RiskMetrics™ Technical Document* (4th ed, J P Morgan/Reuters, New York, 1996), pp 117-121.

Exhibit 16.8—*continued*

Cashflow Map

	Date	Flow	Yield	Present Value	1 Mo.	3 Mo.	6 Mo.	12 Mo.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	7 Yr.	9 Yr.	10 Yr.
			Vertex		1 Mo.	3 Mo.	6 Mo.	12 Mo.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	7 Yr.	9 Yr.	10 Yr.
			Total Vertex Mapping		—	14,437	31,883	58,428	72,567	67,474	64,375	86,288	105,269	330,972	268,733
			Price Volatility			0.0050%	0.0160%	0.0577%	0.1673%	0.2684%	0.3723%	0.4644%	0.6454%	0.8128%	0.8853%
			RiskMetrics™ Vertex VAR		—	1	5	34	121	181	240	401	679	2,690	2,379
					Present Value Amount by Vertex										
1	1-Jun-97	40,000	5.62	39,091.85		14,437	24,654								
2	1-Dec-97	40,000	5.90	37,951.37			7,228	30,723							
3	1-Jun-98	40,000	6.04	36,818.36				23,037	13,781						
4	1-Dec-98	40,000	6.14	35,683.00				4,668	31,015						
5	1-Jun-99	40,000	6.25	34,553.72					22,519	12,035					
6	1-Dec-99	40,000	6.36	33,420.53					5,252	28,169					
7	1-Jun-00	40,000	6.43	32,329.49						21,627	10,703				
8	1-Dec-00	40,000	6.49	31,261.68						5,644	25,618				
9	1-Jun-01	40,000	6.54	30,235.54							21,425	8,811			
10	1-Dec-01	40,000	6.59	29,228.43							6,630	22,598			
11	1-Jun-02	40,000	6.62	28,261.53								24,461	3,801		
12	1-Dec-02	40,000	6.66	27,315.13								16,806	10,509		
13	1-Jun-03	40,000	6.69	26,396.90								9,939	16,458		
14	1-Dec-03	40,000	6.73	25,496.56								3,672	21,824		
15	1-Jun-04	40,000	6.75	24,629.03									22,511	2,118	
16	1-Dec-04	40,000	6.78	23,786.71									15,894	7,892	
17	1-Jun-05	40,000	6.81	22,971.72									9,890	13,082	
18	1-Dec-05	40,000	6.84	22,174.75									4,382	17,792	
19	1-Jun-06	40,000	6.86	21,412.44										20,895	517
20	1-Dec-06	1,040,000	6.88	537,408.01										269,192	268,216
				1,100,426.76											

Source: This calculation uses the models provided as part of J P Morgan's *RiskMetrics™ Technical Document*, Fourth Edition, 1996

The approach to linear fixed income derivatives is very similar. The fundamental technique requires restating the derivative in terms of a combination of underlying fixed interest transactions at different maturities. This approach is usually applied as follows:

- *Futures contracts*—are treated as a borrowing (investment) at one maturity and an offsetting investment (borrowing) at a different (more distant) maturity. The first maturity will usually coincide with the settlement date of the futures contract while the second date will relate to final maturity of the security underlying the futures contract.
- *Forward Rate Agreements* (FRAs)—are treated exactly the same as futures contracts.
- *Interest Rate Swaps*—are decomposed into two separate fixed interest transactions. The fixed leg is treated as a position in a fixed rate bond while the floating rate flows are treated as a position in a floating rate note. The risk of each set of cash flows can then be calculated independently. The fixed leg is valued as a bond. The floating leg will generally trade around the par value with only the current coupon (fixed with reference to the last floating rate set) creating an interest rate exposure which will need to be incorporated.

The credit risk of the underlying instrument will determine the market risk of the derivatives. Government or swap risk factors will be used to determine the risk of a futures contract depending upon the underlying asset of the futures contract. Swap risk factors will generally be used to calculate the risk of FRAs and interest rate swaps.

Exhibit 16.9 sets out an example of the risk decomposition of a FRA.

Exhibit 16.9

Risk Decomposition—Forward Rate Agreement

The attached table shows the process of decomposing an FRA. The transaction details are specified in the section entitled FRA Details. The FRA is decomposed into a borrowing and investment for the relevant maturities. The market value of each cash flow is then calculated using the appropriate discount rate for each maturity. The decomposed cash flows are then utilised to calculate the VAR of the individual cash flows as allocated to individual maturity vertices which are then amalgamated utilising the correlation between the maturity points to calculate the fully diversified VAR of the FRA.

FRA — Cashflow Mapping and Value at Risk Calculation

FRA Details

Date		11-Jan-97
Currency	USD	
First Leg (mths)		3
Second Leg (mths)		9
Amount		10,000,000
3m yield		5.25%
9m yield		5.43%
Basis		360

						RiskMetrics™ Volatilities			VaR Simulation	
Maturity (days)	Cash Flow	Market Rate	Market Value (2)	Vertex	CF Map	Yield (3)	Price (4)	VaR (5)	RiskMetrics™	Correlations
92	−10,000,000	0.0525	−9,867,610	3M	−9,867,610	6.1875	0.008	789.4	1	0.52
181	10,403,421	0.0543	10,126,947	9M	10,126,947	6.5914	0.068	6,886.3	0.52	1
		Total	259,337		259337.116		Diversified	6,510.8		

Source: This calculation uses the models provided as part of J P Morgan, *RiskMetrics™ Technical Document*, (4th ed, 1996).

5.2.4 Risk decomposition—currency

Currency positions are decomposed for risk measurement in terms of the cash flows in the relevant currencies. All currency positions are translated and risk measured in terms of a base functional currency presumed to be the home currency of the relevant entity.

The basic approach is to identify the currency positions in terms of the inflow and outflow in the relevant currency pairings as at the relevant maturity point.

Spot positions are stated as the cash flows receivable and payable and the relevant currency pairing VAR factor can be applied.

Currency derivatives are also decomposed in a similar manner:

- *Currency forwards or futures*—are treated as cash inflows and outflows at a forward maturity date. The position is then decomposed into separate spot currency position (calculated by discounting the cash flows back to the spot date using the swap rates for the relevant maturity) and long and short position in the relevant interest rates in each currency. The currency risk VAR is determined by applying the relevant currency VAR to the spot position while the interest rate risk of the position is calculated by applying the interest rate risk factors. The total risk of the position is given by the individual risk which are then consolidated using the correlation between the currency and two interest rate risk factors.
- *Currency swaps*—are treated as two separate fixed interest transactions in the respective currencies with each security being decomposed into the separate interest rate risk factors in the individual currency. The currency exposure is then incorporated, including the impact of correlation as between the interest rate risk factors and the currency, when each fixed interest bond is translated into the base reporting currency. Where one leg of the currency swap is on a floating rate basis, the approach utilised is identical to that used for the floating rate component of an interest rate swap.

Exhibit 16.10 sets out an example of the risk decomposition of a currency forward.

Exhibit 16.10

Risk Decomposition—Currency Forward

The attached table shows the process of decomposing a currency forward. The transaction details are specified in the section entitled Forward Details. The forward is decomposed into cash flows in the respective currencies at the forward date using the contracted forward date. The market value of each cash flow is then calculated by discounting using the appropriate discount rate in each currency. The calculation isolates the risk of the forward into a spot transaction and the borrowing or investment transaction in the respective currency. The VAR of the individual components are then calculated as the VAR of the spot currency and interest rate element, which are then amalgamated utilising the correlation between the separate risk components to calculate the fully diversified VAR of the currency forward.

Currency Forward — Cashflow Mapping and Value at Risk Calculation

Forward Details

FX Rate	USD/CHF
Spot FX Rate	1.44
Term (Years)	1
USD Yield	6.00%
CHF Yield	2.00%
CHF Face Value	10,000,000
1 day time horizon	

This FX Forward to sell USD against the CHF decomposes into:

1. A Spot FX deal to sell USD and buy CHF
2. A 1 year USD borrowing
3. A 1 year CHF investment

Risk Decomposition

Instrument	Cash Flow	Yield	PV	Forward Rate	Price Volatility	VAR Estimate	Correlation Matrix		
							Spot FX Deal	CHF 1 Year	USD 1 Year
Spot FX Deal			−6,808,279		0.6852	−46,651	1.00000	0.23989	0.01768
CHF 1 Year	10,000,000	2.00%	9,803,922	1.38566	0.0322	2,277	0.23989	1.00000	0.16349
USD 1 Year	−7,216,776	6.00%	−6,808,279		0.0577	−3,927	0.01768	0.16349	1.00000

Undiversified VAR	52,854
Diversified VAR	46,363
Diversification Benefit	6,491

Source: This calculation uses the models provided as part of J P Morgan, *RiskMetrics™ Technical Document*, (4th ed, 1996).

5.2.5 *Risk decomposition—equity*

The risk decomposition of equity transactions will depend on whether it represents a well diversified portfolio of equities which approximates the market index or positions in individual equity securities with a large component of firm specific risk.

This categorisation is related to the risk factors that are applicable. Where the equity positions approximate the risk of the market index, the volatility of the market index is utilised to derive the risk of the transaction. Where the position does not approximate the market index, there are two choices for measuring the risk of the position:

- the market volatility must be adjusted for stock specific risk using the beta of the individual stock; or
- stock specific volatilities must be utilised.

The use of stock beta introduces the problem of basis or correlation risk as between the price changes of the individual stock and the changes in the index.

Irrespective of the risk factors to be used, the decomposition of the position embodies the following approach. Each equity position is restated in terms of the market value of the security as at the specific maturity point. Spot positions are stated as positions in the equity security the risk of which is determined by the application of the market index volatility or the stock specific volatilities. Equity derivatives are also decomposed in a similar manner:

- *Equity forwards or futures*—are treated as a position in the equity as at a forward maturity date. The position is then decomposed into separate spot equity position (calculated by discounting the transaction cash flows back to the spot date using the swap rates for the relevant maturity) and long or short position in the relevant interest rates in each currency. The equity risk VAR is determined by applying the relevant equity VAR to the spot position while the interest rate risk of the position is calculated by applying the interest rate risk factors. The total risk of the position is given by the individual risks which are then consolidated using the correlation between the equity and the interest rate risk factor.
- *Equity swaps*—are treated as two separate transactions; the first being in the equity as at the forward date, and the second being in a floating rate bond. The equity exposure is calculated as a series of forwards and the interest rate exposure on the floating rate leg is calculated using an identical approach to that used for the floating rate component of an interest rate swap. The total risk of the components is consolidated including the impact of correlation as between the interest rate risk factors and the equity.

The process of restating the forward equity position by discounting the future cash flows back to the spot dates requires assumptions to be made regarding the expected dividend income cash flows payable on the security or the portfolio.

Exhibit 16.11 sets out an example of the risk decomposition of an equity position. *Exhibit 16.12* sets out an example of the risk decomposition of an equity forward.

Exhibit 16.11

Risk Decomposition—Equity Position

The attached table shows the process of decomposing an equity position in the spot market. The VAR of equity position is calculated utilising the volatility of *the market index* adjusted by the stock specific risk of the relevant equity security as expressed by the beta (β) of the stock.

Equity Position

Stock Details

Company	GIANT	
Stock Price	$	18.00
Stock Holding		100,000
Present Value		1,800,000
Stock Beta		1.55
Market Volatility		2.21%
95% confidence		3.65%
Holding Period	1 day	

VAR =	Present Value × 95% volatility × Beta
	1,800,000 × .0365 × 1.55
	101,835

Exhibit 16.12

Risk Decomposition—Equity Forward

The attached table shows the process of decomposing an equity forward. The transaction details are specified in the section entitled Stock Details. The forward is decomposed into a position in the equity by calculating the market value of the forward cash flow using the discounted rate for the maturity. The calculation isolates the risk of the forward into a spot transaction and borrowing. The VAR of the individual components are then calculated as the VAR of the spot equity and interest rate element, which are then amalgamated utilising the correlation between the separate risk components to calculate the fully diversified VAR of the equity forward.

Equity Forward Position

Stock Details

Company	GIANT	
Spot Stock Price	$	18.00
Forward Stock Price	$	18.7200
Stock Holding		100,000
Forward Value		1,872,000
Forward Term	1 year	
Stock Lending rate %pa		2.00%
1 year interest rate		6.00%
Stock Beta		1.55
Market Volatility		2.21%
95% confidence volatility		3.65%
Beta Adjusted Volatility		5.66%
Holding Period	1 day	

This FX Purchase of GIANT shares decomposes into:

1. A Spot purchase of GIANT shares
2. A 1 year USD borrowing
3. A 1 year Stock Loan

Exhibit 16.12—*continued*

Risk Decomposition

Instrument	Cash Flow	Yield	PV	Price Volatility	Undiversified VAR	Correlation Matrix: Spot Share Purchase	1 Year Borrowing	1 Year Stock Loan
Spot Share Purchase			−1,800,000	5.6613	−101,903	1.00000	−0.00400	−0.22400
1 Year Borrowing	−1,908,000	6.00%	−1,800,000	0.0322	579	−0.00400	1.00000	0.33240
1 Year Stock Loan	−1,836,000	2.00%	−1,800,000	0.0577	1,038	−0.22400	0.33240	1.00000

Undiversified VAR	103,520
Diversified VAR	102,138
Diversification Benefit	1,382

Source: This calculation uses the models provided as part of J P Morgan, *RiskMetrics™ Technical Document*, (4th ed, 1996).

5.2.6 Risk decomposition—commodities

Commodity positions are decomposed in a manner analogous to that applicable to equities.

Spot positions are stated as positions in the commodity the risk of which is determined by the application of the volatility of the specific commodity or a similar or related commodity.

Commodity derivatives are also decomposed in a similar manner:

- *Commodity forwards or futures*—are treated as a position in the commodity as at a forward maturity date. The position is then decomposed into separate spot commodity position (calculated by discounting the transaction cash flows back to the spot date using the swap rates for the relevant maturity) and long or short position in the relevant interest rates in each currency. The commodity risk VAR is determined by applying the relevant commodity risk factor to the spot position while the interest rate risk of the position is calculated by applying the interest rate risk factors. The total risk of the position is given by the individual risks which are then consolidated using the correlation between the commodity and the interest rate risk factor.
- *Commodity swaps*—are treated as a series of forward contracts on the commodity which are decomposed as above. Alternatively, the swap can be treated as a position in a fixed interest bond (represented by the fixed cash flows to be paid or received calculated as the amount commodity purchased or sold at the agreed forward price) and an opposite position in a floating rate bond (where the cash flows are based on a floating commodity price usually based on the near month futures contract price). The fixed leg can be treated as a fixed interest security while the risk of the floating is analogous to the floating rate note component of the interest rate swap where only the first cash flow is known with a price exposure on the near contract.

The process of restating the forward commodity position by discounting the future cash flows back to the spot dates requires assumptions to be made regarding the convenience yield on the commodity.

5.2.7 Risk decomposition—structured notes

Structured notes can be defined as debt securities with embedded derivative elements. These instruments are decomposed for risk purposes into the individual elements, usually a fixed or floating rate security and the derivative component on the relevant asset, for the purposes of risk decomposition. The individual components are then treated consistent with the framework outlined. *Exhibit 16.13* sets out an example of the risk decomposition of a structured note transaction.

Exhibit 16.13

Risk Decomposition—Structured Note

The attached table sets out the risk decomposition of an inverse floater.[5] The details of the transaction are set out in the section entitled Inverse Floater Details. The inverse floater is broken up into a fixed rate bond and the embedded interest rate swap. The cash flows on the two transactions are then decomposed into a maturity matrix and are then utilised to generate the VAR both on an undiversified and diversified basis (incorporating the correlation between the relevant rates).

Structured Note — Inverse Floater

Calculate the VAR of an inverse floater transaction which pays 12.2%pa minus 6 month LIBOR for a term of 1 year.

Inverse Floater Details

Currency	USD
Face Value	10,000,000
Fixed Rate Receipt	12.20%
Floating Rate Payment	6mth LIBOR
Term	1 Year
Coupon Frequency pa	2
1 Year Bond rate	6.00%
1 Year swap rate	6.20%
6 Month Yield	5.43%
Time Horizon	One Month

This inverse floater can be decomposed into:

1. A 1 Year Fixed rate bond paying 6% pa
2. A 1 Year Receiving Fixed interest rate swap

5. For a detailed coverage of all forms of structured note transactions, see Satyajit Das, *Structured Notes and Derivative Embedded Securities* (Euromoney Publications, London, 1996), Ch 4.

Exhibit 16.13—*continued*

Term Yrs	Bond Cashflows	Swap Fixed Leg	Floating Leg	Net Cashflows	Zero Yield	Price Volatility	Correlations		Present Values	VAR Estimate
0.5	300,000	310,000	−10,000,000	−9,390,000	5.43%	0.1800	1	0.84	−9,141,800	−16,455.2
1	10,300,000	10,310,000		20,610,000	6.21%	0.3900	0.84	1	75,679.3	19,404,952
		Total		11,220,000					10,263,152	

Undiversified VAR	92,135
Diversified VAR	62,498
Diversification Benefit	29,637

Source: This calculation uses the models provided as part of J P Morgan, *RiskMetrics™ Technical Documents* (4th ed, 1996).

The analysis ignores for the sake of convenience the embedded interest rate cap which is incorporated in the structure to prevent the interest rate coupon of the inverse floater becoming negative.

5.3 Application of VAR to options

VAR techniques are well suited to measurement of risk on asset and linear derivatives (forward, futures or swaps). The application of VAR techniques to quantify the risk of options or portfolios containing options presents difficulties. This difficulty arises from a number of sources:

- The price movements of options are non-linear, that is for a given change in the asset price the price change of the option is not constant. This potential acceleration or de-acceleration of the market risk (which is equivalent to the gamma risk of an option) creates difficulties in modelling the exposure of options.
- The impact of changes in volatility on the price of options (the option's vega risk).
- The impact of time decay on the price of the option (the option's theta risk).[6]

In effect, the non linearity of the option price function means that the second parameter required to calculate VAR—the sensitivity of the position to a change in the relevant risk factor—is not constant. This means the VAR estimates may over or underestimate the market risk as it assumes this parameter is constant. The effect of changes in volatility and time decay impact the final term of the VAR calculation as they are additional terms which may impact upon the price of the option and therefore must be incorporated by forecasting potential changes in these factors to derive an accurate measure of risk. Volatility is not a factor in the pricing of assets and forwards and consequently are irrelevant to the quantification of risk of these linear price risk instruments. Theta, in contrast, is not unique to options. The value of both assets (for example, debt instruments) and forwards change as a function of time. However, the degree of change is smaller and the impact on risk less significant.

The difficulties are manifested in two separate contexts: first, where options are included in the portfolio; and secondly, where the portfolio consists of options which are dynamically hedged with positions in the asset (delta hedging). The second position exacerbates the problem of exposure quantification in that changing price exposure of the option requiring re-hedging. This dynamic hedging requirement introduces additional risks such as hedging uncertainties, liquidity constraints, and cashflow constraints. The additional risk impacts on the market risk of the position.

In practice, the difficulties that arise in calculating VAR in the first context are most marked for options with a short time to maturity where the option is trading near to the strike price (near or at-the-money). This holds true where a 1 day VAR estimate is sought to be utilised. Where a longer risk horizon for the calculation of VAR is utilised, the non-linearity of option prices has the potential to distort option prices more significantly, irrespective of the type of option. The problems identified in the second context are present *in all portfolios* where options are dynamically hedged.

6. See discussion in Chapter 10 on option risks.

There are several possible approaches to the calculation of VAR for options. These approaches are used quite consistently across asset classes. These approaches include:

1. The delta based method;
2. The delta method incorporating adjustments for gamma, vega and theta; and
3. Simulation approaches.

The delta based method entails the risk decomposition of the option position using the following steps:

1. The option delta is calculated using an appropriate option pricing model.
2. The delta is used to determine a position in the asset equivalent to the option.
3. The asset position is then included in the appropriate asset class as the relevant risk position.
4. The VAR of the option is calculated as the VAR of the equivalent asset position using standard VAR techniques.

This approach has the advantage of simplicity and ease of computation. However, the approach has a number of deficiencies:

- The non-linearity of the option price is not incorporated and a large change in the asset price will result in a change in the option premium which will vary from the VAR estimate.
- The assumption that the asset volatility remains unchanged. This is because the delta of the option will be volatility specific. Changes in volatility will result in market value changes in the option which are not captured by VAR.
- The delta estimate may be model dependent in that the choice of option model will dictate the delta and consequently the risk.

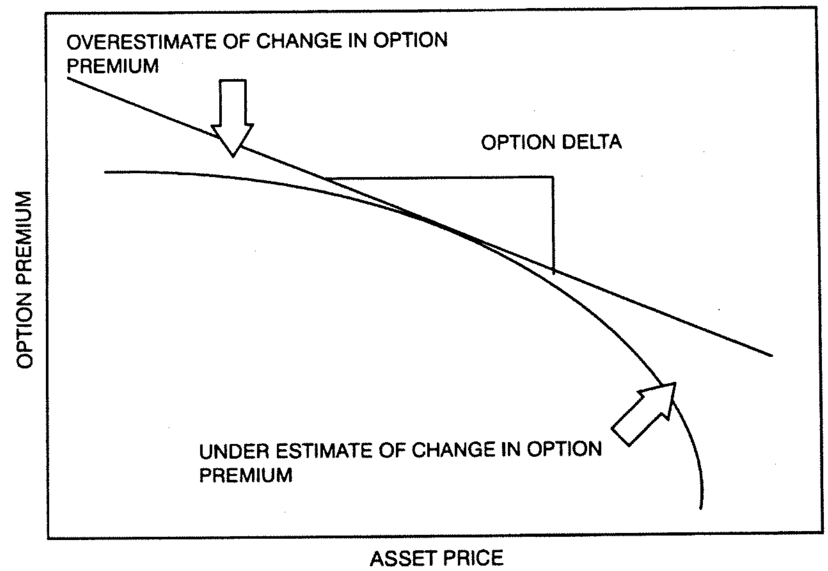
The delta method can be expanded to incorporate the gamma, vega and theta risks of an option.

The gamma risk of an option can be illustrated graphically (see *Exhibit 16.14*). The diagram (depicting the option price function for a sold call option) highlights that the delta based method is inaccurate as it measures the slope of the option price function at a specific point in time at the current market price. The delta based method adjusted for gamma requires the incorporation of an adjustment to compensate for the change in slope of the option price function.[7] This adjustment itself will only be accurate for small changes as the change in curvature of the option price function is not constant. This means that for a large change in the asset price will cause gamma to alter rendering any gamma adjusted VAR measure inaccurate.

7. See J P Morgan/Reuters, op cit n 4, pp 129-133.

Exhibit 16.14

Gamma Risk of Option



The adjustment for VAR for vega requires incorporating the volatility as a separate risk factor. Volatility risk factors of all assets (effectively, *the volatility of volatility*) for separate maturity vertices as well as correlations would need to be incorporated in the risk data set. The volatilities would then be used to determine the exposure to volatility changes.

Theta is incorporated by estimating the forward price of the option.

The adjusted delta technique is particularly difficult where a longer risk horizon is utilised as each of these risk factors is time sensitive and is characterised by significant non-linearity.

The final approach, discussed in greater detail below, is simulation approaches. These approaches entail the use of various approaches, to simulate the performance of the option over the risk horizon based on expected movements in market risk factors, including asset price and volatility. The simulation approach because it incorporates the full revaluation of option positions captures the problems of gamma and vega risk to a greater extent and more accurately than other approaches.

5.4 Data sources/estimating volatilities and correlations

VAR approaches require volatilities for individual risk factors as well as the correlation between risk factors to measure risks. The major criteria for estimating these volatilities and correlations include:

- consistency of methodology;

- computational efficiency; and
- accuracy and robustness of the estimates.

A major consideration in calculating the relevant volatilities and correlation is the quality of the data. Even small errors in the time series can have a significant impact on the volatilities and correlation which can generate inaccuracies in the risk estimates. Key considerations include:

- inaccurate and missing data and the methodology by which it has been adjusted;
- ensuring that the data is consistent; for example, the data should be synchronous, that is it is calculated at the same time, to ensure the validity of the data set.

Data sets are usually obtained from external providers or collected by the entity. The external data source has the advantage of independence while the maintenance of an inhouse data set has the advantage of allowing greater control over the data collected and the quality. The costs of each approach is also a factor.

There are a number of possible approaches to obtaining these estimates:

1. forecasts;
2. implied volatilities and correlations; and
3. historical data.

In practice, historical volatilities are generally used.

Forecasts would entail development of subjective forecasts of the estimated volatilities of individual risk factors and the correlation between them. This approach suffers from the following deficiencies:

- it is subjective; and
- it may be difficult to implement consistently over a large set of required data.

Implied volatilities and correlation would require extracting the required estimates from existing traded instruments. Volatilities would be derived from traded options while correlation estimates would be backed out of exotic options (quanto options, basket options etc where correlation factors are inherent in the pricing). This approach suffers from the following difficulties:

- Data may not be available for all required risk factors. This is particularly the case with correlation estimates. This unavailability reflects the fact that the universe of options, particularly longer dated options and exotic options, may not be extensive enough to provide the required data.
- The data may be affected by the choice of model used to iterate out the volatility or correlation estimate. The liquidity or other institutional factors may also impact on the estimate.
- There is no evidence that implied volatilities and correlations are a better predictor of future actual volatility or correlations than, say, historical estimates.

These factors favour the use of historical volatility and correlation estimates. For example, the BIS Market Risk Guidelines require the use of a trailing 1 year's historical data for VAR calculations.

The process of deriving the required historical volatility and correlation information usually takes one of the following forms:

- the use of historical estimates calculated and supplied by an external party; and
- the maintenance of extensive data sets to calculate the required estimates.

J P Morgan's RiskMetrics™ is an example of the first approach. RiskMetrics™ consists of daily estimates of the volatility and correlations of a large number of rates and prices covering currency, interest rates (government and swap rates), commodity and equity indexes. The second approach requires the entity to store the data or acquire it from a vendor of financial market data and calculate the necessary information itself.

Irrespective of the source of the estimates, the basic procedure for deriving the estimates entails the following steps:

1. Identify the historical data to be utilised.
2. Adjust the data through smoothing or bootstrapping techniques.
3. Calculate the estimates.
4. Calibrate the results obtained.

In practice, usually high frequency data (daily) prices of the relevant risk factors are used. The length of the time series used is subjective but typically between 1 to 3 years data is utilised. The data set is usually a constantly trailing period which is updated regularly (usually no less frequently than quarterly).

The data may need to be adjusted through smoothing or bootstrapping techniques. Smoothing techniques may be used for the following reasons:

- to allocate greater importance to more recent data than older data;
- to filter out the potential impact of certain events; and
- to overcome inadequacies of the available data sets.

The use of smoothing can be illustrated by reference to RiskMetrics™ which uses an exponential moving averages of historical asset price movements as the basis for deriving volatility and correlation estimates. This is done to increase the sensitivity of the estimates to large price movements and subsequent gradual declines in volatilities. Research by J P Morgan undertaken in conjunction with RiskMetrics™ indicates that the smoothed series generally provides more accurate estimates of volatility and correlations.[8]

Alternative forms of smoothing include the use of adaptive techniques such as the Autoregressive Conditional Heteroskedascity (ARCH) type models for estimating volatilities.[9]

8. See discussion in Chapters 8 and 9 of estimation procedures using smoothing techniques.
9. See discussion in Chapter 9 of ARCH type models in estimation of volatility estimates for risk management purposes.

Bootstrapping techniques are generally required for different reasons. They may be useful for allowing estimation from relatively small sample sizes. For example, deriving monthly volatility and correlation estimates from daily volatilities and correlations.

The calculation procedures are relatively straightforward:

- Volatility estimates are calculated as the standard deviation of the (usually) logarithms of the asset price changes.
- Correlations are calculated utilising ordinary least squares or other correlation technique.

Exhibit 16.15 sets out an example of the calculation of volatility and correlation estimates.

Exhibit 16.15—Part One

This exhibit sets out an example of calculating volatility and correlation estimates for the oil price, A$/US$ exchange rate on a monthly basis over ten years. These calculations are used in the case study provided in Appendix A. The table below calculates the parametric 95% confidence loss. The second part of this exhibit calculates the 95% confidence loss using the actual historical distribution.

Time series data

Data Frequency (d,w,m,q)	m	Monthly

		Price/Yield input			Asset Return - % change		Monthly
Name		US$ WTI	AUD/USD	LIBOR	US$ WTI	AUD/USD	LIBOR
Statistics		Absolute Price/Yield Statistics			Asset Return Statistics		
No of obs		121	121	121	120	120	120
Maximum		36.04	0.89	8.82	39.22%	6.17%	11.78%
Minimum		13.77	0.65	2.86	−20.91%	−10.60%	−13.12%
Mean		19.71	0.75	5.46	**0.40%**	**0.15%**	**−0.08%**
Std		3.38	0.05	1.64	**7.56%**	**2.57%**	**4.03%**
95% confidence loss		5.58	0.08	2.71	**12.48%**	**4.24%**	**6.65%**
Kurtosis		7.0797	0.1191	(0.8603)	5.29	2.11	1.03
Skewness		2.0081	0.0629	0.1099	1.00	−0.62	−0.19
		Asset price/Yield			**Calculated % returns**		
Dec-86	1	16.1	0.6648	5.53			
Jan-87	2	18.65	0.6608	5.43	14.70%	−0.60%	−1.82%
Feb-87	3	17.75	0.6748	5.59	−4.95%	2.10%	2.90%
Mar-87	4	18.3	0.7053	5.59	3.05%	4.42%	0.00%
Apr-87	5	18.67	0.7048	5.64	2.00%	−0.07%	0.89%
May-87	6	19.43	0.7137	5.66	3.99%	1.25%	0.35%
Jun-87	7	20.07	0.7203	5.67	3.24%	0.92%	0.18%
Jul-87	8	21.34	0.6978	5.69	6.14%	−3.17%	0.35%
Aug-87	9	20.31	0.7124	6.04	−4.95%	2.07%	5.97%
Sep-87	10	19.53	0.7194	6.4	−3.92%	0.98%	5.79%
Oct-87	11	19.86	0.6757	6.13	1.68%	−6.27%	−4.31%
Nov-87	12	18.85	0.7052	5.69	−5.22%	4.27%	−7.45%
Dec-87	13	17.27	0.7225	5.77	−8.75%	2.42%	1.40%
Jan-88	14	17.13	0.7138	5.81	−0.81%	−1.21%	0.69%
Feb-88	15	16.79	0.7198	5.66	−2.00%	0.84%	−2.62%
Mar-88	16	16.19	0.7388	5.7	−3.64%	2.61%	0.70%
Apr-88	17	17.86	0.7585	5.91	9.82%	2.63%	3.62%
May-88	18	17.42	0.8051	6.26	−2.49%	5.96%	5.75%
Jun-88	19	16.52	0.794	6.46	−5.30%	−1.39%	3.14%
Jul-88	20	15.49	0.8045	6.73	−6.44%	1.31%	4.09%
Aug-88	21	15.52	0.8069	7.06	0.19%	0.30%	4.79%
Sep-88	22	14.53	0.7829	7.24	−6.59%	−3.02%	2.52%
Oct-88	23	13.77	0.8256	7.35	−5.37%	5.31%	1.51%
Nov-88	24	14.14	0.8781	7.76	2.65%	6.17%	5.43%
Dec-88	25	16.38	0.8555	8.07	14.71%	−2.61%	3.92%
Jan-89	26	18.02	0.889	8.27	9.54%	3.84%	2.45%
Feb-89	27	17.93	0.7996	8.53	−0.50%	−10.60%	3.10%
Mar-89	28	19.48	0.8194	8.82	8.29%	2.45%	3.34%
Apr-89	29	21.07	0.7928	8.65	7.85%	−3.30%	−1.95%
May-89	30	20.12	0.7484	8.43	−4.61%	−5.76%	−2.58%
Jun-89	31	20.05	0.7553	8.15	−0.35%	0.92%	−3.38%

Exhibit 16.15—Part One—*continued*

Jul-89	32	19.78	0.7524	7.88	−1.36%	−0.38%	−3.37%
Aug-89	33	18.57	0.7656	7.9	−6.31%	1.74%	0.25%
Sep-89	34	19.59	0.7764	7.75	5.35%	1.40%	−1.92%
Oct-89	35	20.09	0.7831	7.64	2.52%	0.86%	−1.43%
Nov-89	36	19.85	0.7815	7.69	−1.20%	−0.20%	0.65%
Dec-89	37	21.1	0.7927	7.63	6.11%	1.42%	−0.78%
Jan-90	38	22.86	0.7708	7.64	8.01%	−2.80%	0.13%
Feb-90	39	22.11	0.7594	7.74	−3.34%	−1.49%	1.30%
Mar-90	40	20.38	0.7542	7.9	−8.15%	−0.69%	2.05%
Apr-90	41	18.42	0.7509	7.77	−10.11%	−0.44%	−1.66%
May-90	42	18.2	0.7691	7.74	−1.20%	2.39%	−0.39%
Jun-90	43	16.69	0.789	7.73	−8.66%	2.55%	−0.13%
Jul-90	44	18.45	0.7901	7.62	10.03%	0.14%	−1.43%
Aug-90	45	27.31	0.8162	7.45	39.22%	3.25%	−2.26%
Sep-90	46	33.5	0.8265	7.36	20.43%	1.25%	−1.22%
Oct-90	47	36.04	0.7847	7.17	7.31%	−5.19%	−2.62%
Nov-90	48	32.33	0.7745	7.06	−10.86%	−1.31%	−1.55%
Dec-90	49	27.28	0.7733	6.74	−16.98%	−0.16%	−4.64%
Jan-91	50	25.23	0.7849	6.22	−7.81%	−1.49%	−8.03%
Feb-91	51	20.47	0.7851	5.94	−20.91%	−0.03%	−4.61%
Mar-91	52	19.9	0.7752	5.91	−2.82%	−1.27%	−0.51%
Apr-91	53	20.83	0.7817	5.65	4.57%	0.83%	−4.50%
May-91	54	21.23	0.7609	5.46	1.90%	−2.70%	−3.42%
Jun-91	55	20.19	0.7681	5.57	−5.02%	0.94%	1.99%
Jul-91	56	21.4	0.7775	5.58	5.82%	1.22%	0.18%
Aug-91	57	21.69	0.7848	5.33	1.35%	0.93%	−4.58%
Sep-91	58	21.88	0.7995	5.22	0.87%	1.86%	−2.09%
Oct-91	59	23.23	0.7837	4.99	5.99%	−2.00%	−4.51%
Nov-91	60	22.46	0.7848	4.56	−3.37%	0.14%	−9.01%
Dec-91	61	19.49	0.7598	4.07	−14.18%	−3.24%	−11.37%
Jan-92	62	18.78	0.7498	3.8	−3.71%	−1.32%	−6.86%
Feb-92	63	19.01	0.7546	3.84	1.22%	0.64%	1.05%
Mar-92	64	18.92	0.7684	4.04	−0.47%	1.81%	5.08%
Apr-92	65	20.23	0.7593	3.75	6.69%	−1.19%	−7.45%
May-92	66	20.97	0.7589	3.63	3.59%	−0.05%	−3.25%
Jun-92	67	22.38	0.7488	3.66	6.51%	−1.34%	0.82%
Jul-92	68	21.77	0.7442	3.21	−2.76%	−0.62%	−13.12%
Aug-92	69	21.33	0.7134	3.13	−2.04%	−4.23%	−2.52%
Sep-92	70	21.88	0.714	2.91	2.55%	0.08%	−7.29%
Oct-92	71	21.68	0.6954	2.86	−0.92%	−2.64%	−1.73%
Nov-92	72	20.34	0.6823	3.13	−6.38%	−1.90%	9.02%
Dec-92	73	19.41	0.688	3.22	−4.68%	0.83%	2.83%
Jan-93	74	19.03	0.6786	3	−1.98%	−1.38%	−7.08%
Feb-93	75	20.08	0.6957	2.93	5.37%	2.49%	−2.36%
Mar-93	76	20.32	0.7058	2.95	1.19%	1.44%	0.68%
Apr-93	77	20.25	0.7116	2.87	−0.35%	0.82%	−2.75%
May-93	78	19.95	0.6769	2.96	−1.49%	−5.00%	3.09%
Jun-93	79	19.09	0.6722	3.07	−4.41%	−0.70%	3.65%
Jul-93	80	17.77	0.6834	3.04	−7.17%	1.65%	−0.98%
Aug-93	81	17.99	0.6708	3.02	1.23%	−1.86%	−0.66%
Sep-93	82	17.5	0.6453	2.95	−2.76%	−3.88%	−2.35%
Oct-93	83	18.15	0.6661	3.02	3.65%	3.17%	2.35%
Nov-93	84	16.61	0.6586	3.1	−8.87%	−1.13%	2.61%
Dec-93	85	14.51	0.6711	3.06	−13.52%	1.88%	−1.30%

Exhibit 16.15—Part One—*continued*

Jan-94	86	15.03	0.7112	2.98	3.52%	5.80%	−2.65%
Feb-94	87	14.78	0.7178	3.25	−1.68%	0.92%	8.67%
Mar-94	88	14.68	0.7008	3.5	−0.68%	−2.40%	7.41%
Apr-94	89	16.42	0.7124	3.68	11.20%	1.64%	5.01%
May-94	90	17.89	0.7361	4.14	8.57%	3.27%	11.78%
Jun-94	91	19.06	0.7291	4.14	6.34%	−0.96%	0.00%
Jul-94	92	19.65	0.7393	4.33	3.05%	1.39%	4.49%
Aug-94	93	18.38	0.7425	4.48	−6.68%	0.43%	3.41%
Sep-94	94	17.45	0.7393	4.62	−5.19%	−0.43%	3.08%
Oct-94	95	17.72	0.7422	4.95	1.54%	0.39%	6.90%
Nov-94	96	18.17	0.7674	5.29	2.51%	3.34%	6.64%
Dec-94	97	17.16	0.7768	5.6	−5.72%	1.22%	5.69%
Jan-95	98	18.48	0.7583	5.71	7.41%	−2.41%	1.95%
Feb-95	99	18.52	0.7395	5.77	0.22%	−2.51%	1.05%
Mar-95	100	19.18	0.728	5.73	3.50%	−1.57%	−0.70%
Apr-95	101	20.36	0.7299	5.65	5.97%	0.26%	−1.41%
May-95	102	18.88	0.7138	5.67	−7.55%	−2.23%	0.35%
Jun-95	103	17.38	0.7086	5.47	−8.28%	−0.73%	−3.59%
Jul-95	104	17.62	0.7389	5.42	1.37%	4.19%	−0.92%
Aug-95	105	17.89	0.7524	5.4	1.52%	1.81%	−0.37%
Sep-95	106	17.54	0.755	5.28	−1.98%	0.34%	−2.25%
Oct-95	107	17.67	0.7566	5.28	0.74%	0.21%	0.00%
Nov-95	108	18.27	0.7469	5.36	3.34%	−1.29%	1.50%
Dec-95	109	19.54	0.745	5.14	6.72%	−0.25%	−4.19%
Jan-96	110	17.76	0.7447	5	−9.55%	−0.04%	−2.76%
Feb-96	111	19.59	0.7635	4.83	9.81%	2.49%	−3.46%
Mar-96	112	21.43	0.7793	4.96	8.98%	2.05%	2.66%
Apr-96	113	20.95	0.7854	4.99	−2.27%	0.78%	0.60%
May-96	114	19.77	0.7983	5.02	−5.80%	1.63%	0.60%
Jun-96	115	20.92	0.789	5.11	5.65%	−1.17%	1.78%
Jul-96	116	20.45	0.7731	5.17	−2.27%	−2.04%	1.17%
Aug-96	117	22.25	0.7909	5.09	8.84%	2.28%	−1.56%
Sep-96	118	24.2	0.7924	5.15	8.40%	0.19%	1.17%
Oct-96	119	23.25	0.7919	5.01	−4.00%	−0.06%	−2.76%
Nov-96	120	23.7	0.816	5.03	1.92%	3.00%	0.40%
Dec-96	121	25.9	0.7945	5.02	8.88%	−2.67%	−0.20%

Summary Volatility and Correlation Coefficients

Monthly	67% Price Volatility	95% Price Volatility	Correlation Coefficient US$ WTI	AUD/USD	US$ LIBOR
US$ WTI	7.56%	12.48%	1.00	0.09	0.08
AUD/USD	2.57%	4.24%	0.09	1.00	0.16
US$ LIBOR	4.03%	6.65%	0.08	0.16	1.00

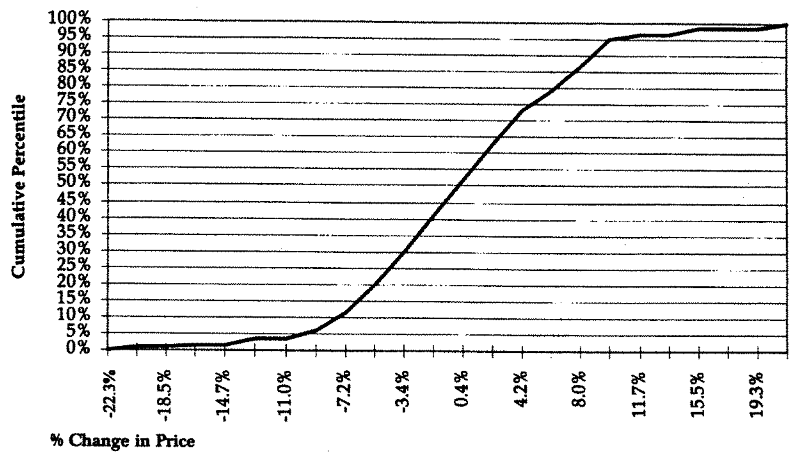
Annualised	67% Price Volatility	95% Price Volatility	Correlation Coefficient US$ WTI	AUD/USD	US$ LIBOR
US$ WTI	26.20%	43.24%	1.00	0.09	0.08
AUD/USD	8,91%	14.69%	0.09	1.00	0.16
US$ LIBOR	13.97%	23.05%	0.08	0.16	1.00

Exhibit 16.15—Part Two

Using the historical time series from the first part of this exhibit the actual 95% confidence level price movement is calculated using the actual cumulative frequency distribution for the three data series. As the table shows, in this instance the parametric VAR over-estimates the 95% confidence (or 5% confidence loss).

	US$ WTI	AUD/USD	LIBOR	
Price Exposure	Fall	Rise	Rise	
5% Percentile Loss Price Change	**−9.69%**	**4.21%**	**6.30%**	See Graphs below
Parametric Result	−12.48%	4.24%	6.65%	
Difference	−2.79%	−0.03%	−0.35%	

Cumulative Price Distribution - WTI



Cumulative Price Distribution - A$/US$

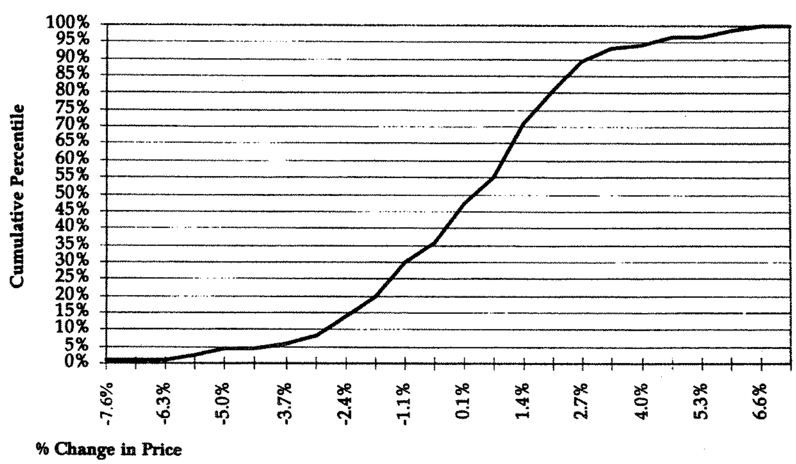
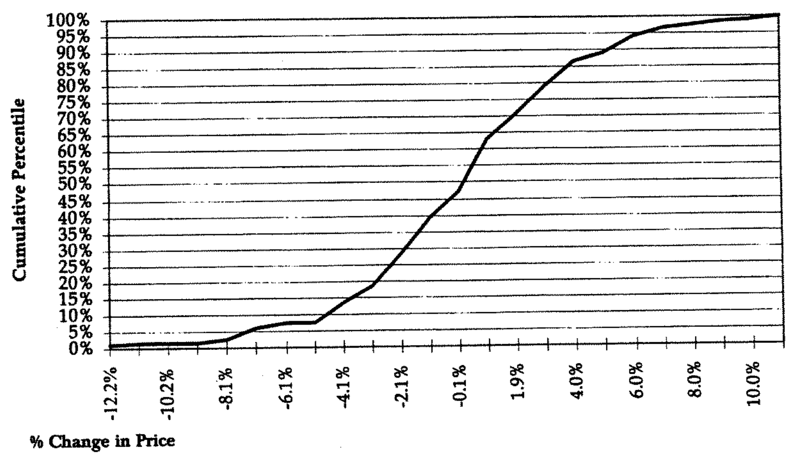


Exhibit 16.15—Part Two—*continued*

Cumulative Price Distribution - LIBOR



The results obtained may be calibrated to test the confidence in the estimates. This is usually done using statistical tests of significance and measuring standard error estimates.[10]

6. TYPES OF VAR

6.1 Technical components of VAR

VAR consists of three elements:

1. The current position in the relevant risk factors.
2. The sensitivity of the position to changes in the risk factor.
3. The forecast adverse change in the risk factors.

The first step is relatively unambiguous being equivalent to the market value of the position calculated at current market prices and rates. The differences in the types of VAR derive from choices in the second and third steps.

The sensitivity of the position can be derived in one of the following ways:

- *Delta based*—calculated as the first derivative of the instrument or portfolio with respect to a small change in the risk factor; for example, the PVBP or DVO1 of a cash flow.
- *Full valuation based*—calculated by recalculating the value of the instrument or portfolio using the normal market accepted valuation model relevant for the instrument.

10. See discussion in Chapter 19 regarding tests of statistical significance.

The forecast of future prices and rates can be generated using one of the following techniques:

- Using implied or historical volatilities and correlations.
- Using actual historical prices and rates.
- Using specified price changes or randomly generated price paths.

In practice, the choices in these two steps can be combined in a number of combinations to derive the measure of risk. There are three primary types of VARs: Analytic VAR; Historical VAR; and, Simulation VAR. Choices within each VAR approach and individual variations used by organisations generate various sub-categories of each type of VAR calculation.

6.2 Analytic VAR

Analytic VAR (also referred to as the variance/co-variance method) entails the use of historical volatilities and correlations to derive the market risk of a portfolio. This approach requires the following steps:

1. Apply a system of risk decomposition to reduce the portfolio to identified risk factors.
2. Obtain the historical volatilities of individual factors and correlations between risk factors from an external source (such as RiskMetrics™) or calculate them directly from historical data.
3. Scale the historical volatility by the number of standard deviations consist with the confidence level desired.
4. Calculate the VAR by applying the volatilities and correlation to the decomposed risk factors.

The techniques described above are an example of Analytic VAR.

In practice, Analytic VAR is generally used with delta based measures of portfolio sensitivity although in theory it is feasible to use full revaluation approaches.

The advantages of Analytic VAR include:

- The elegance and ease of application of the technique.
- Its capacity to be used to analyse complex transaction within a simple framework.
- Computational ease and simplicity, although the size of the correlation matrix can become large and unwieldy.

The disadvantages of Analytic VAR include:

- The need to make a number of statistical assumptions:
  - Assume that the future distribution of changes in the risk factors is normal.
  - Assume stationarity of the volatility and correlation estimates.
- The approach, as already noted, is not capable of accurately capturing the risk of options.
- The need for either historical data sets to generate the volatilities or correlations or availability of external estimates of these parameters.

- It is difficult to incorporate second order risk factors such as gamma (convexity risk), vega (volatility risk) and theta (impact of time decay).
- The process of scaling the risk factor over the risk period by taking the daily VAR estimate and multiplying it by √ risk horizon may not accurately capture the risk.

The process utilised to calculate the VAR under this approach suffers from the problem that a certain amount of information is lost in the computational process. The use of deltas, even for assets or forwards on certain instruments, such as long dated bonds, which have a degree of non-linearity or convexity, means that certain aspects of portfolio risk are over simplified leading to potential inaccuracies in the calculation of VAR.

6.3 Historical VAR

Historical VAR is based on revaluation of the current portfolio using historical rates and prices to arrive at the risk of the positions. Application of this approach requires the following steps:

1. The portfolio is defined in terms of risk factors, either as the instruments or risk factors (using the process of risk decomposition).
2. A historical data set is selected.
3. The historical data set is transformed to the current value date for the calculation of VAR.
4. The portfolio is revalued utilising the normal valuation models/ algorithms based on the historical data sets repeatedly to determine the changes in the portfolio values.
5. The VAR estimate is derived from the set of value changes calculated using either a percentile ranking or using statistical methods.

The Historical VAR approach can be based on the value of the instruments or decomposed risk factors. Under the instrument approach, a historical data set of the mark-to-market prices and rates *for each instrument* is required. Under the risk decomposition approach, each transaction is decomposed into the defined risk factors and mark-to-market values *for the risk factors* is required. In practice, for large portfolios the second approach is utilised. This is because of the lower number of price and rates for risk factors that must be stored and, more importantly, the ability to capture the risk of more complex instruments.

The historical data set will usually cover a nominated number of days, ranging between 90 and 500 days (3 to 24 months). This data is usually derived from a data base consisting of the rates and prices used to revalue the portfolio for mark-to-market purposes which has been stored for reutilisation in estimating VAR.

The data set has to be adjusted to the current value date for the VAR calculation. This requires the historical prices and rates to be related to the current rates and prices. There are a number of choices:

- *Absolute rates and prices*—this would utilise the *absolute* historical prices and rates to revalue the portfolios. The current portfolio is revalued using each day's actual price or rate irrespective of the starting rate or price level.

- *Absolute changes in rates and prices*—this would utilise the absolute price *change* in each rate and price to alter the current price and rates to generate a set of future prices based on the changes of the past to revalue the portfolio.
- *Relative changes in rates and prices*—this is similar to the absolute changes in rates and prices except that the relative or percentage changes (usually calculated as price relatives) of historical prices and rates are used to create the set of future prices.

The use of relative changes in rates and prices is the most logical methodology for the following reasons:

- it is consistent with the no-arbitrage approach to financial markets;
- it avoids the possibility of negative asset prices or rates (possible under the second option); and
- it minimises the impact of absolute price cycles in asset prices and rates and the resultant impact on the risk estimate.

The data set used to revalue can merely be the complete set of transformed historical prices and rates utilised continuously. Under this approach, to estimate risk using back data of 90 days would require only 91 days prices to calculate 90 relative price changes which are used to adjust the current prices and rates to generate the revaluation data set. An alternative methodology would be to generate more than the required set of price changes and randomly sample drawing the required 90 price and rate changes for the available data.

The random sampling technique is clearly more computationally demanding. Where the distribution is stable, the choice of method is unlikely to result in significant differences, at least statistically. However, where the distribution is not stable or where the immediate past is not considered a representative sample period, the sampling method may offer advantages in forecasting risk.

The revaluation is usually done using the conventional valuation models employed, such as cash flow models or actual market prices and options pricing models (closed form, numerical or Monte Carlo Simulations).

The risk value is the change in the value of the portfolio is given by:

$$\Delta PV_t = PV_{t+n} - PV_t$$

Where

ΔPV_t is equal to the projected change in the value of the portfolio at time t

PV_{t+n} is equal to the value of the portfolio at time t + n days

PV_t is equal to the value of the portfolio at time t

t is the VAR value date

n is the risk time horizon (usually given in days)

In practice n would be set to either 1, 10 or for longer risk horizons to say 30 days. Notice that the risk is estimated as the actual movement in the portfolio value over the risk period by calculating the *actual* change in value of the portfolio over the risk period. An alternative may be to capture the *single day VAR* and scale it to the required holding period by multiplying by the square root of the holding period (see discussion above).

The actual VAR estimate is derived from the changes in portfolio value calculated in one of two ways:

- *Percentile ranking method*—this requires ranking the price changes by amount from losses to gains. Once ranked, the VAR measure is calculated as the change in the value of the portfolio which is consistent with the percentile ranking that coincides with the confidence level required. For example, if the Historical VAR calculation is done over 250 days, then the appropriate percentile ranking for 99% (95%) confidence is the lowest 1% (5%) percentile ranking which coincides with the 3rd or 13th largest loss. The VAR estimate is equal to that particular loss figure.
- *Statistical (normal distribution) method*—this entails calculating the sample mean and standard deviation of *the changes in portfolio value*. The VAR is then calculated as the mean of the sample minus the standard deviation scaled by the appropriate number of standard deviations for the confidence level required (1.65 for 95% and 2.33 for 99%). The distribution may be reviewed for skewness and kurtosis prior to ensure normality prior to calculation of the VAR estimate.

Where the distribution is normal, either method will produce similar values, albeit not identical ones. The statistical method has the advantage of improving the prediction if the irregularity of the measure is due to sampling error. In the case of a non-normal distribution, the percentile ranking method is to be preferred.

The Historical VAR approach has significant advantages:

- It is simple and easily comprehensible and communicable.
- The capacity to incorporate actual price series incorporating *actual volatilities and correlations,* including any non-normal characteristics (skews, fat tails etc) of the actual distribution. This avoids the statistical assumptions made under Analytical VAR.
- The ability to use actual valuation models matching risk measurement to portfolio valuation and income determination.
- The inherent capacity to capture the effects of gamma (convexity risk), vega (volatility risk) and theta (impact of time decay).
- Ease of implementation as the revaluation models are already present for valuation purposes and limited data requirements as only historical revaluation data is required.

The disadvantages of Historical VAR include:

- The possibility that the historical period selected is not a good predictor of the risk horizon and creates significant inaccuracies in the risk measure. (It is probable that the same problems would affect the risk measures using Analytic VAR under the same circumstances.)
- The time needed to run the Historical VAR calculation which is a function of the time frame needed to revalue the entire portfolio,

keeping in mind that the portfolio may need to be revalued 250 times to generate the risk estimate.

- The computational requirements may be large although with a risk decomposition approach the demands are likely to be no greater than for Analytic VAR.

6.4 Simulation VAR

The term Simulation VAR is a generic term which encompasses a number of techniques for modelling the performance of a portfolio and deriving risk estimates. The essential components of a simulation approach include:

1. A large set of random paths for prices and rates is generated.
2. The current portfolio is then revalued based on the generated prices and rates *for each path*.
3. The changes in the value of the portfolio are used to derive VAR and other risk measures by either using the percentile ranking method or statistical techniques as described above.

The Historical VAR calculation described above is a limited form of simulation. In practice, the different simulation models are distinguished by differences in the following components:

- the use of either delta-based or full revaluation models to determine the risks of the portfolio; and
- the process adopted to generate the price paths.

The use of full revaluation is favoured because of its ability to capture the full non-linearity of risk as well as second order impact of changes in volatility and time decay. Delta-based valuation is useful because it utilises existing parameters and provides some computational efficiencies. However, the loss of measurement accuracy and the failure to capture the impact of gamma, vega and theta mean that delta-based measures are generally not accurate.

There are two major forms of Simulation VAR techniques—fixed scenarios and Monte Carlo simulation.

Fixed scenario techniques necessitate the current portfolio being revalued following pre-specified changes in all asset prices and rates. The change in portfolio value provides an estimate of risk under the conditions specified. Given that the fixed scenarios are generally constructed to approximate expected *worst case* movements in the relevant risk factors, the portfolio risk measure is assumed to indicate accurately extreme portfolio risk values.

The major benefit of fixed scenario VAR is that it provides a useful stress test for the portfolio under the forecast conditions.

Fixed scenario simulations also have a number of disadvantages:

- The VAR and other risk estimates suffer from the problem that the risk revealed is dictated by the scenario nominated and has no significance other than in that context. To the extent that the fixed scenario over or under estimates the magnitude of movements in the risk factors the risk measures will inaccurately predict the risk profile of the portfolio.

- The performance of the portfolio will depend uniquely on the structure of the current portfolio. Application of the same fixed scenario stress test to *different portfolios* will predictably yield different views of risk. This lack of consistency and comparability is significant.
- The performance of the portfolio assumes no corrective action or adjustment to portfolio positions. In reality, portfolio managers will take corrective action, to the extent possible, as markets move in the manner forecast. The risk measures derived assuming no action may therefore tend to overestimate risk.
- The number of fixed scenarios run will generally be limited thereby limiting the risk estimates obtained which will not allow generation of a complete distribution of outcomes.

These weaknesses mean that the fixed scenario technique while useful in generating stress tests is not useful in deriving VAR measures.

Monte Carlo simulations entail generation of random multiple paths (between 500 and 10,000) of asset prices and rates which are then used to revalue the portfolio to create a distribution of changes in value of the portfolio which are then used to generate the risk statistics.

The major element in Monte Carlo processes is the choice of path generation mechanisms. There are two choices:

1. parametric; and
2. historical.

Parametric Monte Carlo entails the use of historical volatilities and correlations in conjunction with a selected stochastic price or rate process to generate the price paths. Typical approaches include the use of log normal stochastic processes for returns on assets. Mathematical models may be used to generate the future set of risk factors. For example, interest rate term structure models may be used to generate future interest rate paths. *Exhibit 16.16* sets out some examples of stochastic term structure models that are commonly used.[11]

11. Other types of term structure models include Hull-White, Black-Derman-Toy, and Heath-Jarrow-Morton—refer Chapter 6.

Exhibit 16.16
Interest Rate Models

Black Scholes model

The standard Black Scholes relative diffusion model can be used to specific the following stochastic model for interest rates:

$$dr = \mu\, dt + \sigma\, d\, \omega$$

Where

dr is the change in the short term interest rate

μ is a deterministic drift function per unit of time

dt is a short time interval

σ is the volatility of r

d ω is a standard random number generator

Vasichek model

The Vasichek model specifies the following stochastic model for interest rates:

$$dr = \alpha\, (\gamma - r)\, dt + \sigma\, dz$$

Where

dr is the change in the short term interest rate

α is the parameter (greater than 0) which describes the speed at which r revert to a long run average value

γ is the long run value of r

r is the short term interest rate

dt is a short time interval

σ is the volatility of r

dz is a random variable chosen from a normal distribution with mean 0 and variance dt

The process specified identifies that the change in the short term rate r over the interval dt will have two components:

1. A deterministic component (α (γ—r) dt), whereby r will revert to a long run value at a speed parameter (α).
2. A stochastic component (σ dz), which will change randomly.

The structure of the first term implies that if r is close to (away from) its long run value, the deterministic term will be small (large). This term reflects the premise of mean reversion whereby interest rates tend towards some normal rate. The stochastic term will be larger as the time over which change occurs increases. The structure is designed to be consistent with the general pattern of evolution of interest rates in capital markets. The specified process for interest rate changes allows the derivation of valuation formula for a discount bond which in turn facilitates the solution for the value of interest rate derivative products.

Source: O A Vasichek, "An Equilibrium Characterisation of the Term Structure" (1977) 5 *Journal of Financial Economics* 177.

Historical Monte Carlo entails using historical data, rather than mathematical models, to generate the price paths. This approach randomly samples historical price changes (absolute prices, absolute price change, or (the most preferred) relative changes in prices, as described above) to generate the price paths for revaluation.

Common approaches include the use of a nominated stochastic process, which is consistent with valuation approaches, with parameter values being estimated from historical time series.

It is possible to distinguish between two types of Monte Carlo simulations:

1. *Passive*—where the underlying prices and rates follows the selected paths but there is no intervention.
2. *Active*—where the portfolio is rehedged to remain delta neutral as prices and rates alter.

Passive simulation is of limited usefulness in measuring risk, particularly, in portfolios with significant optionality. Active simulation is more valuable for analysis of dynamic trading issues, particularly, in portfolios containing option positions hedged in the asset market (see discussion above). An additional advantage of dynamic trading is the reduction in dependence upon secondary factors (such as drift terms) in the stochastic models used to generate the price paths.[12]

The advantages of Monte Carlo approaches include:

- The large number of price paths and consequently distributions of risk measures generated provide more reliable and comprehensive measures of risk.
- It explicitly captures the convexity of non-linear instruments as well as changes in volatility and time.
- The ability to use full valuation allows greater accuracy in deriving the risk estimates.
- The approach is very flexible allowing development of a wide range of price paths which may enable a fuller view of the risk to be modelled.

The disadvantages of Monte Carlo simulation approaches include:

- The dependence on the accuracy of either the stochastic process specified and/or the historical data used to generate the price path for the validity of the risk estimates.
- The computational requirements which can be formidable and the consequent lack of speed of generating VAR estimates (calculation can take up to several hours). This has cost implications for the user.

6.5 Types of VAR—comparison

The various approaches to VAR are clearly different and not readily reconcilable. The differences as between the approaches predicatably result in different risk estimates.[13] The problem in practice is that a choice has to be made in respect of the approach to be adopted in any particular context or within any organisation.

12. See discussion in Chapter 17 of using Monte Carlo simulations to measure option risks.
13. See Tanya Styblo Beder, "VAR: Seductive But Dangerous" 1995 (September-October) *Financial Analyst's Journal* 12; J V Jordan, and R J Mackay, "Assessing Value At Risk For Equity Portfolios: Implementing Alternative Techniques" in Rod Beckstrom, Alyce Campbell, and Frank Fabozzi (eds), *Handbook Of Firmwide Risk Management* (1996).

The choice between the three general types of VAR is dictated by a consideration of a mixture of the following factors:

1. Coverage of instruments.
2. Accuracy and tractability of risk measures, including statistical assumptions underlying approach.
3. Implementation requirements covering valuation models, risk decomposition and data requirements.
4. Systems requirements covering information technology issues.
5. Ease of communicability of concept and results to users.

Coverage issues focus on the capacity of the approach to include all traded instruments and the treatment of options.

Two approaches are evident in respect of the first issue: focus on instruments or the use of risk decomposition to identify and aggregate the risk factors irrespective of the instrument. The first approach requires price, volatility and correlation information to be maintained *on every instrument.* The second approach has the advantage of allowing economies to be achieved in respect of the number of risk factors and thereby improves computational efficiency as well as the capacity to handle newer innovations within the risk factor framework.

Analytic VAR uses risk decomposition and therefore is capable of handling most traded instruments. Historical VAR and Simulation VAR can be undertaken using either instruments price values or risk factors. In practice, the second is favoured for the reasons identified above.

The treatment of options is more problematic. The major issues relate to the additional risks introduced through the inclusion of non-linear risk in the portfolio. Analytic VAR may generate significant inaccuracies in measuring risk because of its weaknesses in capturing the gamma, vega and theta risk of options. Historical and Simulation VARs, particularly, where full valuation models are used and the portfolio is aged to capture time decay (theta) effects, is better at capturing the additional risk of optionality in portfolios. *Exhibit 16.17* sets out a comparison of Analytic and Simulation VAR approaches in deriving risk measures for a portfolio containing options.

Exhibit 16.17

Measuring the Risk of Option Positions—Analytic VAR vs Simulation Approaches

Assumptions

This example compares the differences in risk estimates obtained from measuring risk through analytic VAR as against simulation approaches for a portfolio containing options.

Assume an investor holds the following positions:

- Investment (long) of US$ 1,000,000 (FFR 4,855,000 at US$/FFR spot rate of US$1: FFR4.855) equivalent of French Franc ("FFR") 2 year zero coupon OAT bond at a yield of 7.147%.
- Purchase of 5 week FFR put/US$ call on FFR 4,228,117/US$ 871,044 based on a strike set at the US$/FFR 5 week forward rate of US$1: FFR 4.864.

The investor's base currency is US$ and the investment risk horizon is 5 weeks.

The investor's risk is measured using three approaches:

1. Analytic VAR based on the option's delta.
2. Analytic VAR incorporating gamma changes using estimation procedures.
3. Monte Carlo simulation.

Analytic VAR—Delta Based

Under this approach, the risk is measured using the sensitivity of the individual positions (the option position being approximated by its delta) and using volatility and correlation estimates derived from the RiskMetrics™ data set.

The results are set out below. The undiversified VAR (for a 5 week holding period and for a 95% (1.65 σ) confidence level) is:

Risk Factor	Position (US$)	Volatility (%)	VAR (US$)
2 year FFR OAT zero	871,044	1.22	10,602
US$/FFR	871,044	5.38	20,550

The option VAR reflects the option delta of approximately 0.4386.

Incorporating a correlation between the risk factors of -0.291, the diversified VAR can be estimated as: US$ 20,197.

The relationship between the value of the currency option and the US$/FFR rate is set out in the following graph:

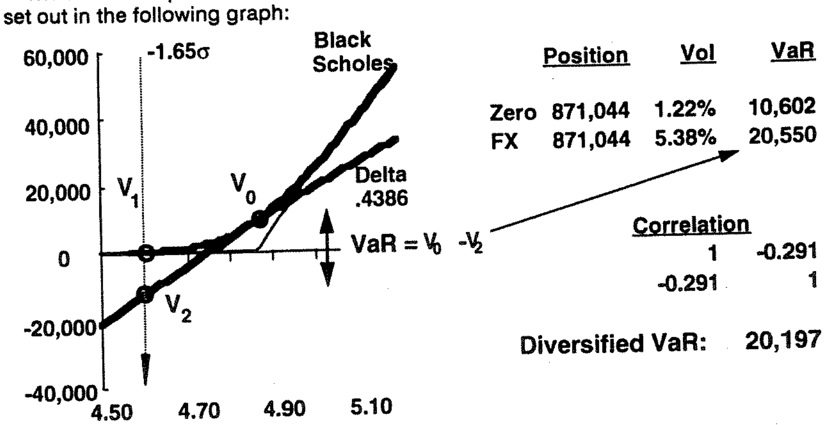


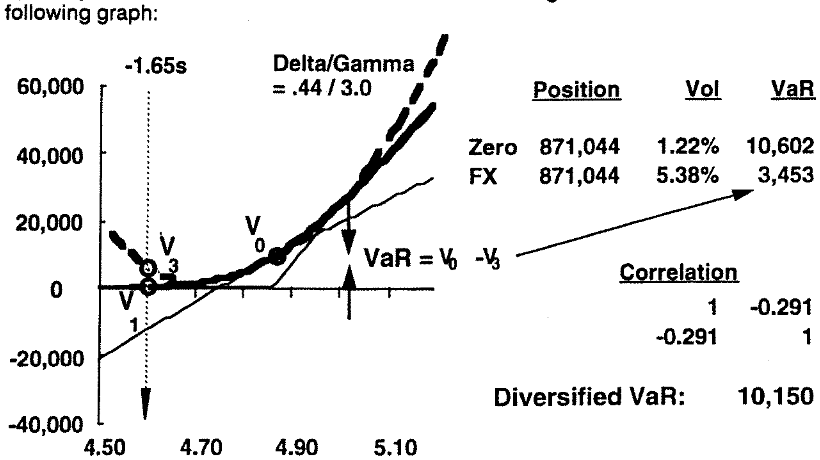
Exhibit 16.17—*continued*

The delta is the approximation of the change in value of the option for changes in the underlying currency measured as the slope of the tangent taken at the current spot rate.

The delta method potentially under or over estimates the change in value of the option. For example, if the spot rate moves from its current level (US$1: FFR 4.85) to, say, US$1: FFR 4.59 (a 1.65 σ change), the value of the option moves from point V0 to V1 in the graph (the option value is calculated using a Black Scholes model). The use of delta would estimate a change in the value of the option from V0 to V2—a much larger fall in value. This would result in an over estimation of the loss. In other words, the undiversified loss VAR estimate of the option of US$20,550 in this case represents an over estimate of the loss.

Analytic VAR—incorporating gamma

The deficiencies of a delta-based methodology for estimation of risk can be overcome by incorporating gamma. This would entail incorporating the option's value by using a non-linear function of the level of the exchange rate. This is set out in the following graph:



The undiversified VAR (for a 5 week holding period and for a 95% (1.65 σ) confidence level) is:

Risk Factor	Position (US$)	Volatility (%)	VAR (US$)
2 year FFR OAT zero	871,044	1.22	10,602
US$/FFR	871,044	5.38	3,453

Incorporating a correlation between the risk factors of -0.291, the diversified VAR can be estimated as: US$ 10,150.

As is evident, the VAR estimate *of the currency option* decreases from US$ 20,550 to US$ 3,453.

The use of the delta-gamma non linear function is also not strictly accurate as it under estimates the real change in the value of the option as measured by revaluation of the option using a Black-Scholes model for large changes in the exchange rate. In this case, the estimated risk is too low. The option *real* VAR using full revaluation is US$ 7,405. The underestimation of risk results from the fact that the curvature of the option value function around the current spot price (which is technically what gamma represents) does not accurately represent the curvature of the value function *at other exchange rates*.

Exhibit 16.17—*continued*

Monte Carlo simulation

To estimate the risk using Monte Carlo simulations, it is necessary to generate a substantial number of paths of interest rate and currency value changes. In this case, some 10,000 paths are created. The paths are created consistent with the current volatility and correlation estimates:

FRF 2 year zero volatility: 9.63%

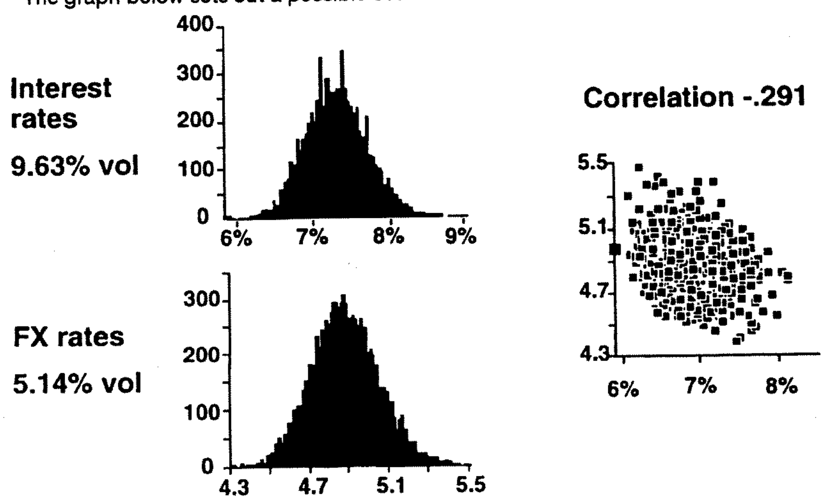
US$/FFR volatility: 5.14%

Correlation: -0.291

The scenario generation requires a number of steps:

1. Decompose co-variance matrix Σ, so that $\Sigma = A{*}A'$.
2. With 10,000 random numbers X generate 10,000 scenarios of returns $Y = A'{*}X$.
3. From 10,000 scenarios of returns compute 10,000 scenarios of rate levels around current forwards $Z = F * \exp(Y)$.

The graph below sets out a possible set of scenarios:

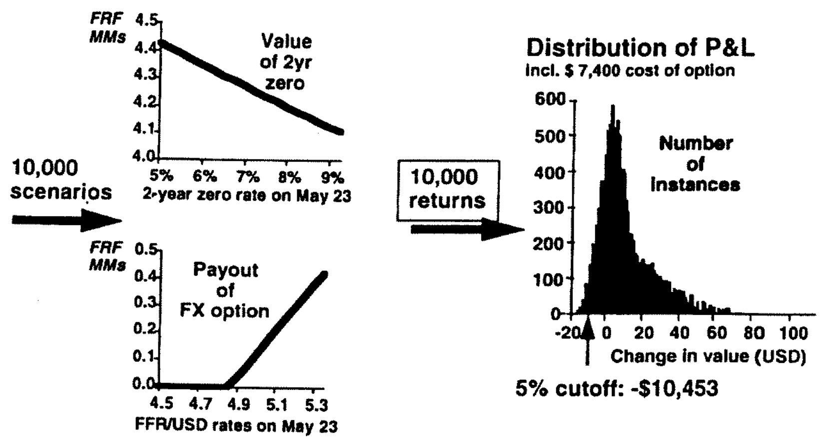


The set of scenarios generated will differ each time the simulation is run although their parameters (volatilities and correlations) are constant.

Once the scenarios are generated, the interest rates and currency values derived are used to revalue the position in each of the 10,000 paths created. The resulting profit and losses are then combined in a histogram. The histogram is then used to calculate the 5% risk estimate.

Exhibit 16.17—*continued*

The graph below sets out the value of the position using the full revaluation method:



In this case, the simulation indicates that over a 5 week risk holding period and using a confidence level of 95%, the expected loss would not exceed US$ 10,453—the effective VAR of the position.

Summary

The VAR risk estimates generated are as follows:

VAR Technique	**VAR Measure (US$)**
Analytic VAR—Delta based	20,197
Analytic VAR—Delta Gamma Based	10,150
Monte Carlo Simulation	10,453

As is evident, the delta-based Analytic VAR estimate is flawed because of its inability to capture the non-linearity of the option. The gamma adjusted measure overcomes this problem partially but may incorrectly estimate risk depending on the magnitude of the price moves. The Monte Carlo simulation provides a more robust risk estimate, albeit at greater cost and slower speed, but is subject to the accuracy of the underlying volatility and correlation parameters used to generate the price paths.

Source: J P Morgan, *RiskMetric™—Technical Document* (3rd ed, J P Morgan, New York, (1995), pp 30-37, as presented by Debra Robertson (J P Morgan Australia Limited, "Fixed Income Research"), presentation to AIC Trading Risk Management Workshop, 24 May 1996.

A central criteria in selection is the accuracy and tractability of the risk measures derived from the model. The accuracy of any model is partly derived from the realisation of the assumptions underlying VAR generally, in particular, the representativeness of the historical data or simulation method or stability of estimates (see discussion in full below). However, in general, the following comments are warranted:

- Analytic VAR will provide reasonable VAR measures where the portfolio does not contain significant optionality, although there is dependence on the distribution of actual market outcomes relative to the assumed distribution.

- Historical VAR and Simulation VAR will provide robust estimates, irrespective of whether the portfolio contains significant optionality, provided either the historical period is representative or the stochastic process used to generate simulation paths is reasonably tractable and sufficient numbers of paths are used.

The major issue is the assumptions underlying Analytic VAR of a normal distribution for past asset returns. The assumption which is discussed more fully below is an important aspect of the Analytic VAR approach and non-realisation of the assumption may significantly mis-estimate risk. The emergence of evidence that markets are not consistent with normal or lognormal stochastic processes mean that extreme outliers may occur significantly more often than predicted by a normal distribution. The problem in this regard is that while markets *on the whole* may be consistent with normal distribution it is these outliers that create the most significant levels of risk in the portfolio. Consequently, failure to adequately capture them may reduce the value of the risk measures *in the situation where they are most needed.*

An advantage of historical simulations is that no distribution need be assumed as the *actual* historical price series is used to directly revalue the portfolio. This removes any risk of masking any skew or kurtosis in the distribution. This adds to the robustness of the risk estimates.

Simulation VAR, particularly, that using Monte Carlo techniques, can, depending on its specification avoid the assumption of a normal distribution. However, to aggregate and consolidate individual VAR of the risk factors may require an assumption of normality although this too can be avoided by rerunning the simulations for the total portfolio.

The implementation measures relate to valuation models, risk decomposition and data requirements.

As noted above, there is a choice between delta based valuation methods (Analytic VAR) and full revaluation methodologies using normal pricing models (generally, Historical VAR and Simulation VAR). The simplicity of delta based models is offset by the difficulties in capturing second order (that is, non-asset price) related risks. However, full revaluation is favoured as it overcomes the identified weaknesses and provides considerably greater richness in the risk estimates.

The process of risk decomposition is essentially a concession to data handling efficiencies as noted above. The process has the disadvantage that the process of decomposition, in particular, the decomposition of cash flow to capture interest rate risk into a limited number of maturity vertices, may reduce the accuracy of the risk measure.

The process of risk decomposition is essential to Analytic VAR. It is optional where Historical or Simulation VAR is utilised. However, in practice, the advantages of using a standard set of risk factors rather than the full set of instruments for the reasons already noted favours using risk decomposition methods for both Historical VAR or Simulation VAR without loss of significant precision of the risk estimates.

The data requirements for any type of VAR is significant. Analytic VAR requires volatilities for each risk factor and correlation as between risk factors. This can be either derived from external sources (such as the J P Morgan RiskMetrics™ data set) or internally based on the required data time series.

Historical VAR requires the relevant price and asset parameters for either each instrument or more realistically each risk factor. While this data requirement may appear formidable, in practice, the major proportion of data is available as it is required to mark portfolios to market. As a consequence, ensuring revaluation prices and data is stored in a form facilitating retrieval for VAR calculations reduces the difficulty in obtaining the data required. The internal data may be required to be supplemented by externally acquired data for new instruments or risk factors where the requisite price history is not internally available.

Historical VAR because of its reliance on actual data does not require the extensive computation of volatilities and correlations required under Analytic VAR. This reflects the fact that the volatilities and correlations are implicit in the data set. The data set for Historical VAR requires transforming to the value date for VAR as described above.

The data requirement for Monte Carlo simulations is varied and will relate to the historical market data or correlations or volatilities required by the stochastic processes assumed.

The systems requirements are varied. The requirements relate to the generation of the projected rates and prices, the risk decomposition and finally the generation of the risk measure.

Analytic VAR places considerable demands on systems in the terms of generation the volatility and correlation estimates, the risk decomposition and also the process of VAR calculation, particularly, the aggregation and consolidation of risk within and across asset classes.

Historical VAR is relatively straightforward. It requires merely limited manipulation of the data to rebase it. The risk decomposition (if utilised) will be identical to that required under Analytic VAR. The calculation of VAR requires repeated revaluations of the portfolio, using existing valuation models from which the VAR estimates can be derived.

Simulation VAR entailing Monte Carlo simulations requires the greatest systems resources. The substantial computing demands of generating and running a significant number of price paths makes this technique both slow and expensive. The data requirements and risk decomposition demands (if applicable) are not significantly different to that for Historical or Analytic VAR.

The ease of comprehension and communicability varies significantly as between the methods. The Historical VAR technique is significantly better on this criteria. It is relatively easily understood and its simplicity and relative transparency and flexibility make it a favoured methodology for communicating with less sophisticated users.[14]

Analytic VAR and Simulation VAR are relatively more difficult to communicate because of the inherent statistical techniques involved and the

14. For example, both Merrill Lynch and J P Morgan have used versions of this approach to communicate information about risk to their shareholders.

requirement of an understanding of these to appreciate the use of these models.

The 1996 BIS Market Risk Guidelines do not recommend a standard methodology from estimating market risk. The BIS recommended the use of both qualitative and quantitative standards for internal bank risk management models. The internal VAR based risk models utilised by regulated financial institutions must meet the following general criteria:

1. Daily computation of VAR measures for all trading positions.
2. A confidence level of 99% (2.33 σ) to be utilised.
3. Minimum holding period of 10 business days.
4. Minimum effective historical observation period of 1 year (250 business days) with the data sets to be updated at least quarterly.
5. The models used should capture the non-linear risk of option instruments.
6. Consolidation to be allowed to incorporate both correlations within risk factor categories and between asset classes.

The BIS Guidelines do not recommend any particular form of VAR model. The BIS Guidelines do, however, recommend the use of separate stress testing to measure risk that might not be captured by the models. The BIS also recommends back testing to test and improve risk management systems.

Exhibit 16.8 summarises the differences between the BIS (Basel Committee) proposal with alternative risk management frameworks (such as RiskMetrics™).

Exhibit 16.18

Comparing the Basel Committee Proposal with RiskMetrics

Issue	Basel Committee Proposal	RiskMetrics
Mapping: how positions are described in summary form	• Fixed income: at least 6 time buckets, differentiate government yield curves and spread curves. • Equities: country indices, individual stocks on basis of beta equivalent. • Commodities: to be included, not specified how.	• Fixed Income: data for 7-10 buckets of government yield curves in 16 markets, 4 buckets money market rates in 27 markets, 4-6 buckets in swap rates in 18 markets. • Equities: country indices in 27 markets, individual stocks on beta (correction for non-systematic risk). • Commodities: 80 volatility series on 11 commodities (spot and term).
Volatility: how statistics of future price movement are estimated	• Volatility expressed in standard deviation of normal distribution proxy for daily historical observations year or more back. Equal weights or alternative weighting scheme provided effective observation period is at least one year. • Estimate updated at least quarterly.	• Volatility expressed in standard deviation of normal distribution proxy for exponentially weighted daily historical observations with decay factors of .94 (for trading, 74 day cutoff 1%) and .97 (for investing, 151 day cutoff at 1%). • Special Regulatory Data Set, incorporating Basel Committee 1-year moving average assumption. • Estimates updated daily.
Adversity: size of adverse move in terms of normal distribution	• Minimum adverse move expected to happen with probability of 1% (2.32 standard deviations) over 10 business days. Permission to use daily statistics scaled up with square root of 10(3.1). Equivalent to 7.3 daily standard deviations.	• For trading: minimum adverse move expected to happen with probability of 5% (1.65 standard deviation) over 1 business day. • For investment: minimum adverse move expected to happen with probability of 5% (1.65 standard deviation) over 25 business days.

Exhibit 16.18—*continued*

Issue	Basel Committee Proposal	RiskMetrics
Options: treatment of time value and non-linearity	• Risk estimate must consider effect of non-linear price movement (gamma-effect). • Risk estimate must include effect of changes in implied volatilities (vega-effect).	• Non-linear price movement can be estimated analytically (delta-gamma) or under simulation approach. Simulation scenarios to be generated from estimated volatilities and correlations. • Estimates of volatilities of implied volatilities currently not provided, thus limited coverage of options risk.
Correlation: how risks are aggregated	• Portfolio effect can be considered within asset classes (Fixed Income, Equity, Commodity, FX). Use of correlations across asset classes subject to regulatory approval. • Correlations estimated with equally weighted daily data for more than one year.	• Full portfolio effect considered across all possible parameter combinations. • Correlations estimated using exponentially weighted daily historical observations with decay factors of 0.94 (for trading, 74 day cutoff 1%) and 0.97 (for investing, 151 day cutoff at 1%).
Residuals: treatment of instrument specific risks	• Instrument specific risks not covered by standard maps should be estimated. • Capital requirements at least equal to 50% of charge calculated under standard methodology.	• Does not deal with specific risks not covered in standard maps.

Source: J P Morgan/Reuters RiskMetrics™—Technical Document Fourth Edition (1996, J P Morgan/Reuter, New York) at p 39

In practice, the various advantages and disadvantages of the individual techniques are to some extent contextual; that is they depend on the type of portfolio and/or the purpose for which risk is measured. However, based on the analysis set out above, the favoured method of VAR estimation is Historical VAR, supplemented by stress testing using Monte Carlo Simulation. The major advantages include:

- The simplicity of the approach.
- The lack of assumptions regarding the distribution of asset price changes and the capacity to subsume intrinsic shifts in volatilities and correlations embedded in the actual data.

- The limited manipulation of the data utilised and the relative ease of data capture and maintenance.
- The use of full revaluation using normal valuation algorithms which capture second order risks such as gamma, vega and theta.
- The ability to capture non-linear risk of optionality efficiently.
- The flexibility of the approach (where risk decomposition is incorporated) to accommodate all instruments and be adapted readily to new instruments.
- The flexibility of deriving the VAR estimate using a choice of method.
- The lower systems demands of this approach.
- The ease of comprehension and communicability of this technique.

When supplemented with regular Monte Carlo simulations, particularly active Monte Carlo techniques, to stress test the portfolio, the Historical VAR approach provide robust risk estimates. The periodic use of Monte Carlo minimises the cost and speed constraints of this technique but allows it to be used to provide additional insight into portfolio behaviour such as dynamic changes in risk as it is rehedged in a portfolio with significant option contents, the impact of shifting volatility, trading liquidity constraints, cash flow impacts and other dynamic risk aspects.

6. ISSUES IN USING VALUE AT RISK

6.1 Assumptions underlying VAR

The VAR methodology requires a number of assumptions about the functioning of capital markets. In practice, the most important assumption relates to:

- the estimation interval or time period; and
- the distribution assumptions.

Both have been alluded to earlier in various contexts. The assumption regarding estimation interval affect *all* VAR methodologies. The distribution assumption affects Analytic VAR and may affect Simulation VAR depending upon the type of stochastic process to generate and calibrate the revaluation price paths.

The estimation time period issues arise because all VAR approaches are based on historical information about asset prices and rates albeit in different ways. Analytic VAR uses volatilities and correlations derived from historical information. Historical VAR uses the historical price series directly to generate the revaluation parameters. Simulation VAR, depending on the type of simulation engine, uses historical data indirectly or directly to generate the price paths or determine parameters for the stochastic models utilised.

To the extent that the historical data used is not representative of the actual period in the future, the risk estimates would be inaccurate. The extent of any inaccuracy and its impact would be dependent upon the extent of the variation. The stability of the estimates used to determine risk are very important.

The stability of J P Morgan's RiskMetrics™ volatility and correlation estimates has been tested.[15] The analysis concluded that these estimates based on historical data weighted using an exponential weighting scheme were reasonably stable but were to varying degree time varying.

The use of historical data forces consideration of the estimation interval to be used to minimise estimation error on projecting risk parameters. This question is similar to the issues arising in relation to selection of a time interval to compute historical volatility for derivation of volatilities for option pricing.

There are two choices:

- *Short*—this uses the most recent data and makes estimates responsive to the most recent movements in market prices.
- *Longer*—this uses longer time periods but is subject to the problems that it is not responsive to recent trends, structural changes in markets and any mean reversion tendency make dominate the estimate.

An additional problem is that even with reasonable samples of time series data, it is unlikely that there are sufficient degrees of freedom, from a statistical viewpoint, to accurately discriminate between individual sources of risk. This problem will be greater the smaller the data series used to derive the risk estimates.

The distributional assumption, which affects Analytic VAR most directly, relate to a number of issues including:

1. Normality—that is, are prices changes in financial assets normally distributed.
2. Mean Change—that is, should the volatility be calculated as the difference between zero and the current deviation from the sample mean.
3. Log versus percentage changes—that is, should volatility estimates be based on the distribution of percentage changes or the logarithms of the changes of asset prices.

Each of these issues have been addressed by J P Morgan in the context of its RiskMetrics™ approach which is predicated on the assumption of normality.[16]

The assumption of a normal distribution of asset price changes provides important advantages in allowing predictions about expected price changes. The RiskMetrics™ research focused on establishing the difference between the observed and predicted frequencies and values of observations in the tails of the normal distribution. This methodology, which used a variety of statistical techniques was designed to provide information on both how frequently extreme outcomes *actually* occurred relative to the *predicted* outcomes as well as how *large* these values were relative to the predicted values.

15. See J P Morgan, *Five Question About RiskMetrics™*; (Morgan Guaranty Trust Company; New York, 1995); J P Morgan/Reuters, op cit n 4, Appendix A.
16. Ibid.

The analysis concluded that the observed frequencies and points were not inconsistent with the assumptions of a normal distribution. The most significant exception was money market or short term interest rates which generally did not exhibit the behaviour consistent with normal distributions. The behaviour of money market rates is quite predictable given the impact of intervention by monetary authorities in the setting of these rates which results in non-random changes in value.

Statistically, standard deviation (volatility) measures the dispersion of price changes around a specified mean. This mean is usually assumed to be zero. An alternative may be to utilise the estimate of sample mean or a conditional zero mean return.

The RiskMetrics™ research concludes that for short (1 day) risk horizons the difference between the zero mean and the estimated mean is not significant but for longer risk horizons (1 month) there are larger differences particularly for money market rates. This means that the zero mean estimator is generally unbiased and viable.

The use of logarithm (compounded) price changes are common in finance as they allow continuous time generalisations of discrete time results and returns for in excess of one day are a simple function of the single day return. The alternative is to use simple percentage change returns. The RiskMetrics™ research concludes that the volatilities and correlations calculated under either type of return while different are not significantly so. The original version of RiskMetrics™ utilised percentage returns but subsequently changed to logarithmic returns.

The tractability of the assumptions should be treated with caution. The results of any tests may be contingent upon the test data and different data sets may give rise to different conclusions. The major problem from a practical viewpoint is that even if markets are consistent with normal distribution assumptions *generally* a small departure (which may or may not be statistically significant) from normality has significant implications for risk management as it may lead to an underestimation of risk which in combination with the structure of the portfolio at a given time may lead to unacceptable and unanticipated losses.

The most significant of the above assumptions is that relating to the normality of the distribution of price changes. Given the growing evidence that the behaviour of risk factors do not conform to normally distributed stochastic processes the risk of an outlying event, which may occur more frequently than standard risk models predicated on normality of distributions anticipate, may in fact be the *primary focus* of risk estimation.

The most tractable and useful estimate of risk is one that most comprehensively captures the complexity of the underlying distribution of changes in risk factors, the interrelationship between risk factors, and their impact on individual portfolio structures. In this regard, the absence of significant assumptions makes the Historical VAR approach attractive.

Where distributional assumptions are required, the use of backtesting techniques to review and calibrate risk modelling techniques is essential.

6.2 Backtesting techniques[17]

The concept of backtesting is essential to the process of evaluating and calibrating risk measurement models. The basic concept is to compare the *actual* observed change in the value of the portfolio with the *risk estimate* provided by the VAR calculated. The essential element is to measure the accuracy of the model prediction against actual changes in portfolio value and to ensure that the model estimates the risk consistent with the desired confidence level.

The key steps in backtesting are as follows:

1. VAR estimates using the relevant VAR model are generated and stored.
2. Actual portfolio profits and losses are calculated using normal mark to market procedures and stored.
3. Periodically, the actual daily mark to market gain or loss is compared to the daily VAR measures (the BIS Guidelines recommend quarterly backtesting using a trailing 250 day (1 year) period).
4. The error fraction (or exceptions) is then calculated as the number of occasions on which the actual trading result exceeded the VAR risk measure.

The VAR estimates should be significantly larger than the trading outcomes for all but a small number of days (for example, at a confidence level of 99%, using a test sample of 250 days (1 year), the error fraction should, intuitively, be around 2-3). To the extent that the error fraction is within or outside acceptable ranges determines the validity of the risk model.

In practice, the process of backtesting is complicated by a number of issues, including:

- the problem of contamination of the portfolio; and
- the problem of fee income or other earnings on the portfolio.

The problem of contamination relates to the fact that for the backtest to have validity the actual trading outcomes must not be contaminated by changes in portfolio composition. This means that the portfolio must be held constant during the relevant period.

The problem of portfolio earnings is related to the issue of contamination. Depending on the risk horizon, the portfolio may generate significant earning which may partially offset trading gains and losses. However, the earning may contaminate the backtest and render more complex the interpretation of the results.

The problems of contamination and the inclusion or otherwise of portfolio earnings increases in importance as the risk horizon utilised is longer. This is because the likelihood of changes in portfolio composition and the impact of earning increases.

The guidelines for backtesting suggested by the BIS favour uncontaminated backtesting and non-inclusion of earnings.

17. For an excellent discussion of backtesting, see Basle Committee on Banking Supervision, "Supervisory Framework For The Use of 'Backtesting' In Conjunction With Internal Models Approach To Market Risk Capital Requirements", January 1996.

Evaluation of the backtest can be informal or formal, based on statistics. The Supervisory approach recommended by the BIS is a good example of the latter.

The BIS approach classifies the result into three colour zones (green, yellow, red) based on the error fraction. *Exhibit 16.19* sets out the BIS model.

Exhibit 16.19

BIS Three Zone Approach to Backtesting Interpretation

ZONE	NUMBER OF EXCEPTION
Green	0
	1
	2
	3
	4
Yellow	5
	6
	7
	8
	9
Red	10 or more

Under the model approach, the backtest is interpreted as follows:

- *Green zone*—the backtest does not suggest a problem with the quality and accuracy of the model.
- *Yellow zone*—the backtest results are not conclusive.
- *Red zone*—the backtest results indicate a problem with the quality and accuracy of the model.

The supervisory response to the backtest lies in the adjustment to the scaling factor for capital required to be held against market risk. The base level of this factor is 3 (that is, the capital required to be held equates to three times the VAR estimate). Depending on the results of the backtest, the supervisory authority may increase the factor, at their discretion. *Exhibit 16.20* sets the increase in scaling factors recommended. For example, where the backtest results are in the red zone, the supervisor would be able to increase the multiplication factor applicable to the model by 1 (increasing it from 3 to 4 times—an effective penalty in capital terms of $33^1/_3$%).

Exhibit 16.20

BIS Three Zone Approach to Backtesting Interpretation—Increase in Scaling Factor

ZONE	NUMBER OF EXCEPTION	INCREASE IN SCALING FACTOR
	0	0.00
Green	1	0.00
	2	0.00
	3	0.00
	4	0:00
	5	0.40
	6	0.50
Yellow	7	0.65
	8	0.75
	9	0.85
Red	10 or more	1.00

The major interpretative problems of a backtest are where the results are inconclusive, that is they are in the yellow zone. The results may be generated by any one of the following factors:

- *Model integrity*—this includes incorrect position or incorrect volatilities or correlations.
- *Model risk factors*—this includes insufficient specification of the risk factors; eg insufficient number of maturity vertices; specific risk (stock specific exposure in equities or credit risk in bonds etc).
- *Market condition*—this covers the occurrence of a low probability event (a stock market crash) or a shift in market volatility levels.
- *Intra-day trading*—this covers income events such as large intra-day changes in the positions.

The analysis of the backtest may be able to isolate the cause of the failure. The second and third factors are potentially the least serious as these events may be expected to occur on some occasions. The first and third are potentially more serious as they point to more fundamental deficiencies in the integrity of the risk measurement process. The supervisory response may incorporate consideration of the likely cause as well as the extent to which the trading outcomes exceeded the VAR measure.

The use of a consistent and regular backtesting protocol is essential to use VAR type model to both measure the accuracy and validity of the risk model as well as allow enhancement and improvements in the basic model.

6.3 Adjustments to VAR methodology

The basic VAR model identified deals adequately with certain elements of risk. It is designed to and fundamentally addresses market risk. However, the basic VAR model provides only limited assistance in measuring liquidity risk and specific risk. It also embodies certain weaknesses in its handling of commodity price risk. In this Section certain enhancements to the basic VAR model designed to address these weakness are considered.

6.3.1 Adjusting VAR for liquidity risk

The concept of liquidity risk encompasses several categories of exposure:

1. *Liquidation risk*—this relates to the fact that a position may be large *relative to the trading liquidity* in that particular security or asset which might impede the elimination of the position by trading.
2. *Trading liquidity risk*—this relates to the volume of trading in a particular asset that might be required, for example to delta hedge a portfolio if the market price moves by a large amount, and the potential risk that the trading volume required may not be able to be executed or can only be executed at larger costs.
3. *Cash flow risk*—this relates to the cash impact of market price movements, for example, in terms of mark-to-market losses on futures positions, and the cash requirements to fund the positions.

Two and 3 are, in reality, extremely important. The best mechanism for quantifying the risk is through simulation methodologies as discussed above.

In contrast, liquidation risk can be encompassed within the VAR model framework. This is because VAR through its risk horizon specifically assumes a liquidation period. A 1 or 10 day risk horizon implicitly assumes that the position held can be eliminated or neutralised *in a period not exceeding the risk horizon*.

In reality, any portfolio will contain a large number of positions with *varying* liquidation periods. Assuming a 10 day risk horizon for risk measurement, a portfolio will generally include a large number of positions which will be able to be adjusted well within that time horizon while there will be other position in less liquid assets which will require a longer liquidation period. This means that the VAR measure may overestimate the risk on liquid positions while underestimating the risk on illiquid positions and the validity of the VAR measure will be contingent on the relative size of the positions.

This problem can be dealt with by amending the basic VAR framework as follows:

- *The holding period adjustment*—this would entail including a time to liquidation concept whereby the greater of 10 days (the minimum risk horizon) and the actual expected liquidation period could be used to derive VAR. The actual liquidation period could be specified as the size of the position divided by a certain percentage of the daily trading volume to ensure the incorporation of the relative position size in the calculation.

- *The use of bid/offer spread adjustment*—this assumes that the relative liquidity of an asset should be reflected in the bid offer spreads for that asset. Amending VAR by incorporating a bid/offer spread component which is variable depending upon the asset may allow the liquidation risk impact on VAR to be covered.

The above methodologies are relatively crude attempts to address the liquidation risk problem. A more sophisticated technique tries to capture the actual behaviour of traders and institutions in seeking to minimise the loss in closing down risk positions.[18]

The basic model respecifies the liquidation horizon for any position (equating to the VAR risk horizon) as that appropriate to minimise the cost of liquidation. The cost of liquidation is specified as the following:

- *Transaction costs*—covering the bid/offer spread adjusted for the size of the position.
- *Cost of exposure*—covering the cost of capital that must be held against the position until the position is liquidated and any cost of hedging the position until liquidation.

Provided appropriate functions for transaction costs and the cost of exposure can be specified, the optimal liquidation horizon is calculated as the period which minimises the identified costs.[19] The identified liquidation horizon is then used as the relevant risk horizon to derive the VAR risk estimate.

Major advantages of this approach include:

- The explicit linkage created between liquidity risk, transaction cost and risk horizon in deriving risk estimates.
- The capture of the tradeoff between transaction cost and cost of maintaining exposure.
- The attempt to capture explicit trader behaviour and risk management practice.

The difficulties with this approach include:

- Problems in accurately specifying the functions for transaction costs and the cost of exposure.
- Changes in market and trading conditions which may require frequent adjustments in these functions.
- Increased computational requirements and demands.
- Added complexity in the risk measures.

18. The liquidity adjusted VAR approach is set out in Dr Colin Lawrence, Gary Robinson and Matthew Stiles, "Incorporating Liquidity Into The Risk Measurement Framework" (1996) 6 *Financial Derivatives and Risk Management* 24; Dr Colin Lawrence and Gary Robinson, "Value At Risk: Addressing Liquidity And Volatility Risks" (1996) 7 *Capital Market Strategies* 24.
19. For a discussion of the process of specifying the appropriate functions see the references cited in the previous footnote.

6.3.2 *VAR and specific risk*

The VAR methodology facilitates the capture, measurement and display of market risk in terms of specified risk factors. The risk factors usually are designed to and do in reality capture, reasonably efficiently, *general* market risk. The risk factors are relatively poor at capturing *specific* market risk.

General market risk, in this context, refers to marketwide movements, that is changes affecting *all instruments*. Specific market risk refers to changes which are unique to and affect *individual or specific instruments*. For example, in interest rate risk terms, general market risk refers to changes in general rate levels, embodied in changes in risk free government rates which affects the total universe of fixed interest securities while specific risk refers to changes in the spread relative to the risk free government rate for a specific issuer which affects the value of fixed interest securities issued by the particular issuer.

In practice, the issue of specific risk is evident in two contexts:

1. individual equity securities; and
2. non-government fixed interest securities.

The specific risk relating to individual equity securities only arises where equities are mapped to the relevant domestic market index and the position does not constitute a well diversified equity portfolio. As noted above, in such a case, individual risk factors *for the individual equity securities* are recommended.

The problem of specific risk in non-government fixed interest securities relates to the fact that only two zero rate curves are utilised for risk measurement—a risk free government curve and the swap curve. This assumes that the relationship between the swap curve and the yields on other non-government (and therefore risky) fixed interest securities is constant. In practice, these spreads can be volatile resulting in the residual or specific risk being substantial.

This is particularly the case with non-investment grade securities whose yields are subject to considerable spread volatility. The empirical research[20] indicates that the correlation between the returns on corporate debt and US Treasury yields decreases as the credit quality, as measured by rating

20. See Richard Bookstaber and David P Jacob, "The Composite Hedge: Controlling The Credit Risk Of High Yield Bonds" (1986) (March/April) *Financial Analysts Journal* 25; Robin Grieves, "Hedging Corporate Bond Portfolios" (Summer 1986) *The Journal of Portfolio Management* 23; Murali Ramaswani, "Hedging The Equity Risk Of High Yield Bonds" (1991) (Sept/Oct) *Financial Analysts Journal* 41.

parameters, decreases. This decrease in correlation to risk free debt returns is paralleled by an *increase* in the correlation between the bonds and equity of the issuer as the credit quality declines. The relationship can be seen from a comparison of the correlations (see *Exhibit 16.21*).[21] Under these conditions, the lack of incorporation of specific risk can lead to significant weaknesses in the risk measures derived.

Exhibit 16.21

Correlation Between Corporate Bonds, Treasury Bonds and Equity

Rating Level	Corelation With Treasury Bonds	Correlation With Equity
Aaa–A	0.86	0.09
Baa–Ba	0.77	0.25
B–Caa	0.51	0.28

Source: Richard Bookstaber and David P Jacob, "The Composite Hedge: Controlling The Credit Risk Of High Yield Bonds" (1986) (March/April) *Financial Analysts Journal* 25 at 26.

In practice, the problem does not lend itself to easy solution as the incorporation of specific risk factors, say based on credit spreads for different rating categories, while desirable from a model validity viewpoint would substantially increase the number of volatilities and correlations. However, in practice, for entities with large positions in securities or instruments with substantial specific risk factors, the incorporation of specific risk into the VAR model is essential.

6.3.3 VAR and commodity yield risk

The problem of commodity risk relates to practical problems of estimating certain price attributes such as storage costs and the commodity convenience yield[22] (effectively, the asset return earned from ownership of or holding the asset).

In theory, the relevant risk factor for commodity transactions would be to decompose all positions in the spot asset and positions in interest rates at the

21. The underlying logic of this relationship is based on the theoretical model which specifies that all corporate securities are claims on the value of the firm. Equity being characterised as a residual claim akin to a call option on the net asset value of the firm (ie assets net of liabilities). Using put call parity, this means that corporate debt equates to the security combined with the sale of a put option structure on the assets of the firm. This analysis is fundamental to the derivation of default risk discussed in detail below. For more detailed discussion of this approach see F Black, and M Scholes, "The Pricing of Options and Corporate Liabilities" (1973) 81 *Journal of Political Economy* 637; R Merton, "On The Option Pricing Of Corporate Debt: The Risk Structure Of Interest Rates" (1974) 29 *Journal of Finance* 449; R Geske, "The Valuation Of Corporate Liabilities as Compound Options" (1977) *Journal of Financial and Quantitative Analysis* 541.
22. See Satyajit Das, "Commodity Swaps: Forward March" (1993) *Risk* 6(2) 41.

relevant maturity vertices in the commodity currency (generally, US$). However, in practice, VAR calculations utilise commodity futures traded on exchanges as the relevant price information and the relevant risk factors.[23]

This reflects a number of factors:

- The fact that commodity transactions, particularly commodity derivatives, such as commodity forwards/swaps etc, are settled against the near or second month futures contract rather than the spot commodity.
- The participation of producers, consumers *and investors* in the commodity futures market which gives this market segment added liquidity and efficiency.
- The higher transparency of the commodity futures markets.

Under this approach, commodity positions are decomposed into equivalent futures positions and the risk factors applied are those related to the futures contracts. The fact that futures contracts have non-constant and decreasing maturities is handled by using a number of adjustment algorithms including:[24]

- Rolling to nearest futures contract (that is the contract which expires closest to a fixed maturity).
- Linear interpolation between the prices of two futures contracts that are adjacent to the relevant fixed maturity.

The methodology identified is generally satisfactory but has a number of deficiencies:

- The liquidity of commodity futures markets, particularly for the very short or longer maturities, can be low.
- The volatility of futures contracts with short terms to maturity may decrease as liquidity declines reflecting the absence of trading interest, illiquidity, and physical delivery concerns.
- The lack of proper capture of the volatility of convenience yields.

These deficiencies may reduce the validity and accuracy of the VAR estimates derived for commodity transactions. In practice, for active commodity traders, it may be necessary to refine the risk factors to encompass these risk aspects to more completely capture the risk of commodity positions.

23. RiskMetrics™ follows this convention for all commodities except gold bullion.
24. These algorithms are suggested by RiskMetrics™.

7. APPLICATIONS OF VAR

7.1 Overview

VAR measures of market risk provide a basis for the management of financial risk in both financial institutions and non-financial institutions. In the first section, applications of VAR to financial institutions are considered while in the following section application for non-financial institutions are examined.

In considering the application of VAR approaches to risk measurement two matters should be noted:

1. VAR approaches are *a component* of an overall risk management framework.
2. VAR approaches measure market risk and other risks (such as credit risk, liquidity and operational risk) which may or may not be fully captured by VAR will need to be captured in developing a *complete* risk profile of an entity.

7.2 Market risk management in financial institution

7.2.1 Range of applications

Financial institutions can be taken to encompass two specific types of institutions: dealer and non-dealer financial institutions and investment institutions. Both groups have exposures to asset price exposures as a result of trading positions or investments which can be measured using VAR approaches. While there may be differences in risk horizons or some other aspects in the use of VAR, the principal elements in the application of VAR are similar.

The use of VAR in market risk management in financial institutions will generally encompass the following applications:

1. Trading risk control and risk management.
2. Performance evaluation.
3. Capital allocation.

VAR may also form the basis of the communication of market risk incurred by an organisation in its activities, for example, in the form of disclosure in financial statements or in investor relations more generally.

7.2.2 Trading risk management

The use of VAR to quantify the market risk in a portfolio is one of the most important applications of this technique. The VAR estimate concisely and precisely, within the range of assumptions made summarises the risk of the positions held at a given point of time resulting from an expected large move in market prices. The ability of VAR to capture and consolidate risk across different asset classes and types of activities (proprietary trading, market-making, underwriting, and investment as well as structural exposure embedded in asset-liability balance sheet mismatches) and communicate *the* *total risk on a firmwide basis* in the form of a *single* $ figure is amongst the most compelling advantages of VAR approaches.

Reports such as the legendary 4:15 PM Report to the Chairman of J P Morgan summarising the total daily earning at risk (DEAR) have enshrined this application of VAR.

This use of VAR also implies the use of VAR based trading limits which are used to control risk taking to acceptable levels within the firm in terms of its capital resources and risk appetite. VAR based limits may still require translation into notional face value amounts for ease of on desk monitoring and trader convenience. VAR based approaches can be used at multiple levels within a trading operation—typical levels would include trader/dealer level, trading desk, asset class or business group, and firmwide. This system of multiple levels would embody the progressive benefits of diversification of risks as the risk is consolidated at increasingly higher levels encompassing greater ranges of activities.

The application of VAR in trading risk control and risk management can also be extended to assisting in enhancing hedging of trading exposures. The use of VAR as the basis for limits and quantification of risk also implies the use of VAR reduction as the basis for optimising hedging behaviour. This approach would seek to reduce risk by entry into transaction which on a portfolio basis reduces the overall VAR or risk of the portfolio. This type of hedging (referred to as correlation based hedges) can overcome deficiencies in the hedge market (for example, because of liquidity, institutional, or regulatory considerations) as well as reducing the transaction costs of the hedge. This contrasts with specific hedges which would be designed to offset *individual* risk positions which may have lower efficiency or higher costs in effecting an equivalent reduction in risk.

7.2.3 Performance evaluation

The ability to quantify the risk of trading positions is essential as both a mechanism for controlling risk as well as aligning risk reward relationships within organisations involved in market risk activities. This entails implementation of risk adjusted performance measurement (RAPM) processes designed to quantify return scaled to the risk incurred in generating that return. RAPM processes are discussed in detail in a following section.

In brief, the linking of return to risk and therefore to the cost of risk capital is essential to the following:

1. The evaluation of individual trading strategies.
2. The evaluation of performance at every level ranging from individual dealer to business.
3. The creation of links between compensation and risk adjusted returns.

These steps are essential to properly aligning the behaviour of traders and business managers with those of shareholders.

7.2.4 Capital allocation

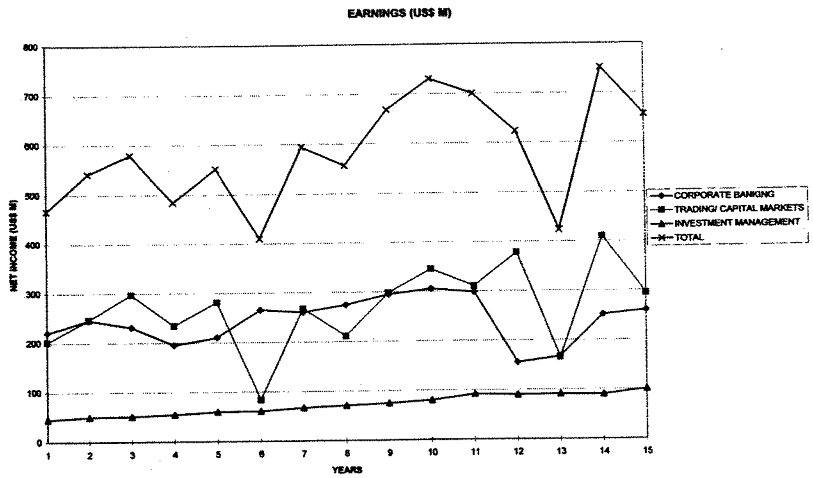
The development of RAPM systems .facilitates the evaluation of both products and businesses in terms of their risk adjusted returns. The analysis of risk adjusted returns will take the form of evaluating returns from individual activities and comparing them to the firm's cost of capital. Where the return is lower than the cost of capital, the product or activity should be discontinued or re-positioned to avoid loss of shareholder value. Products or activities returning above the firm's cost of capital will attract risk capital to enhance shareholder value.

The process of capital allocation can be extended using earning volatilities to cover investment in particular lines of business and to determine the amount of capital that must be held against the risk of earning volatility allowing the quantification of the benefit of operating diversified sets of activities within the same financial institutions. *Exhibit 16.22* sets out a simple example of this approach.

Exhibit 16.22

Capital Allocation—Applications of VAR[25]

The use of VAR in capital allocation in financial institutions can be illustrated using the following example. Assume a bank has the historical earning record set out in the following graph:



The earnings are disaggregated into three operating divisions: corporate banking, trading/ capital markets, and investment management.[26]

The overall volatility of earning is summarised in the following table:

BUSINESS UNIT	VOLATILITY (US$ millions)
Corporate Banking	46
Trading/ Capital Markets	83
Investment Management	18
Total	107

25. The approach used here is similar to that outlined in Chris Matten, "Risk Adjusted Performance Measurement" (1996) 6 *Financial Derivatives and Risk Management* 37-43; see also Chris Matten, *Managing Bank Capital: Capital Allocation and Performance Measurement* (John Wiley & Sons, Chichester,1990).
26. The earning analysis could be disaggregated even further into more specific or narrowly focused business units in the same way.

Exhibit 16.22—*continued*

Volatility is measured as the standard deviation of unit earning in US$ million. The volatility of the earnings of the banks is less than the sum of the individual volatilities because of the fact that the earnings of individual units is not perfectly correlated. The correlation of the earnings of the individual units and the impact of the correlations (effectively, the impact of diversification) is summarised in the following table:

BUSINESS UNIT	UNDIVERSIFIED VOLATILITY (US$ millions)	CORRELATIONS	DIVERSIFIED VOLATILITY (US$ millions)[27]
Corporate Banking	46	0.51	23
Trading/Capital Markets	83	0.89	74
Investment Management	18	0.54	10
Diversification adjustment			40
Total	147	1.00	107

As is evident the overall volatility of earning is lower than the component businesses reflecting the less than perfect correlation between the activities.

This analysis can be used for two primary purposes:

1. The calculation of the volatility of earnings and the capital requirements of the bank.
2. The allocation of capital to individual businesses.

Assume the bank is forecasting earnings of US$ 700 million in the next year. The historical volatility of earning can be used to estimate the potential volatility of earnings. Based on 3 standard deviations (99% confidence levels), it is possible to project that earnings will be:

US$ 700 million ± 3 times US$ 107 million or US$ 379 to US$ 1,021 million.

The analysis shows the potential earning volatility and reveals that the bank's earning at risk are US$ 321 million at the nominated 99% confidence level. Based on this type of analysis the bank's management can adjust the risk profile of the bank to bring it into line with levels considered acceptable to shareholders in one of two ways:

1. Reduce the risk profile of the bank by reducing involvement in more risky or more volatile activities (such as trading).
2. Maintain risk capital to offset the earning at risk. For example, to offset the earnings at risk of US$ 321, assuming a risk free rate of 7.00% pa, the amount of risk capital required to be held equals US$ 4,586 million (calculated as the amount of capital invested at the risk free rate which provides earning of US$ 321 million).

This model also lends itself to determining the amount of capital to be allocated to individual activities. This allocation should be based not on the volatility of the earnings of the individual activity but the *marginal contribution* to volatility of the bank's overall earnings. This approach, which is consistent with overall portfolio theory, is based on the fact that a volatile business activity if imperfectly correlated with the earnings of the bank's other businesses may contribute to a reduction in the bank's overall risk as represented by the volatility of the bank's earnings.

27. The diversified volatility is calculated, approximately, by multiplying the correlation (between the earnings of individual unit and the total earnings of the bank) by the original volatility.

Exhibit 16.22—*continued*

The marginal volatility contribution of each business activity is summarised in the following table:

BUSINESS UNIT	UNDIVERSIFIED VOLATILITY (US$ millions)	DIVERSIFIED VOLATILITY (US$ millions)	MARGINAL VOLATILITY (US$ millions)[28]
Corporate Banking	46	23	15
Trading/ Capital Markets	83	74	56
Investment Management	18	10	9
Diversification adjustment		40	66
Total	147	107	81

The performance of each business unit would then be assessed against its marginal contribution to the overall earnings volatility with capital being allocated to each unit based on its marginal risk contribution.

While theoretically elegant, this approach obviously is difficult to implement in practice:

- The definition of earning is ambiguous. The attribution of earnings for individual business units is difficult and may be arbitrary.
- The method is historical oriented and examines the volatility and correlations that have historically existed. Changes in these volatilities and correlations, as with any VAR analysis, would significantly bias results.
- The distribution of earnings is unlikely to be normal and this creates statistical complications in using this type of approach.

Despite the identified difficulties this approach has considerable benefits in allowing analysis of the risk of individual activities and as a basis for assessing the marginal risk contribution of individual activities as a basis for capital allocation.

7.2.5 Application differences as between dealer financial institutions and investment institutions

The application of VAR as between dealer financial institutions and investment institutions derives largely from the fact that the risk assumed in the former is optional and therefore capable of elimination, while for the latter group is structural and therefore not capable of ready elimination. This flows from the requirement of the investment institution to stay invested at all times.

This difference dictates that the use of VAR as a mechanism for risk management or performance evaluation needs to be amended. The form of amendment would generally take the form of looking at the VAR of a model or benchmark portfolio; this would be the model portfolio prescribed for the asset manager. The VAR measured would be the *incremental risk* of the portfolio assumed as a result of changes in portfolio composition *away from*

28. The marginal volatility contribution is calculated by recalculating the volatility of the total bank earnings excluding each business in turn and taking the difference between the volatility of bank earnings with the business included and excluding the business.

the model portfolio. In essence, the VAR risk estimate would be used to measure the change in risk profile engendered as a result of the investment decisions implemented.

Once this benchmark concept is introduced, the use of VAR as a measure of risk, systems of risk limits, evaluation of optimal hedging strategies and performance evaluation is possible in a manner which is perfectly consistent with the use of VAR in dealer financial institutions.

The concept of capital allocation is also different when applied to investment institutions. The use of risk adjusted returns can aid asset allocation decisions. This may again be allied to model portfolios with the expected returns relative to the VAR risk of a strategy arising from a shift away from the model portfolio composition being utilised to evaluate changes in investment strategy.

A subsidiary problem in application of VAR to investment institutions is the appropriate risk horizon. A short risk horizon (1 day or 10 days) may be appropriate for a dealer financial institution. However, longer time horizons are necessary for investment institutions; for example J P Morgan suggest a risk horizon of 5 weeks.

7.2.6 Market risk management in financial institutions—case study

An example of the application of VAR techniques in the management of risk in a financial institution is set out in *Appendix A* to this Chapter.

7.3 Market risk management in non-financial institutions

7.3.1 Range of applications

Application of VAR techniques to market risk management in non-financial institutions is differentiated by the various factors identified above which distinguishes risk management in non-financial institutions from that in financial institutions. The use of VAR in non-financial institutions is not well developed (see *Exhibit 16.23* which summarises a recent poll of American CFOs).

Exhibit 16.22—*continued*

The marginal volatility contribution of each business activity is summarised in the following table:

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Exhibit 16.23

Corporate Application of VAR Techniques—US Survey Results

Does your company employ VAR?	
Yes	18.0%
No	82.0%
If you do not yet use VAR measurements, have you:	
Considered it?	67.6%
Ruled it out?	32.4%
Does your company use the J P Morgan RiskMetrics™ model?	
Yes	5.3%
No	94.7%
If your firm uses some form of VAR, what does it measure?	
Firmwide risks	40.0%
Investment portfolio	48.0%
Derivatives	36.0%
Interest rate, stock, bond and commodity movements	60.0%
Other	4.0%

Source: Institutional Investor (1996).

In practice, VAR can be applied in non-financial institutions for a number of different applications:

1. Trading risk management where treasuries of industrial corporations operate as profit centres trading for profit in a manner analogous to banks and dealers.
2. Measurement of *financial risk* (such as interest rate exposure on liabilities or investments, currency risk and commodity price risk) and the analysis of efficient hedging alternatives.
3. The measurement of corporate risk from changes in market factors within an integrated risk framework.

The first two applications are fairly restrictive applications which are not dissimilar to the applications in financial institution identified above. The final application is by far potentially the most interesting and valuable use of VAR in corporate context. This application of VAR requires adjustments to the basic VAR framework to expand it into the concept of cash flow VAR.

7.3.2 Concept of cash flow VAR

The concept of cash flow VAR translates the impact of market prices into its impact on an entity's underlying cash flow.[29] The approach is predicated on the fact that for a non-financial corporation cash flow is the major management variable. This reflects the following factors:

- Cash flow discounted to the valuation date at an appropriate discount rate (the cost of capital) is the value of the firm. Consequently, changes in cash flow will have manifest valuation effects.
- Sufficiency of cash flow to meet fixed payments (fixed operating costs as well as fixed debt servicing—interest and principal) is essential as failure may trigger financial distress.

Underlying this approach is the fact that in non-financial institutions, risk is a consequence of operations and the focus of risk management is action to manage the impact of changes in *cash flow*. This is consistent, as noted above, with the cash flow based approaches to shareholder value which are predicated on enhancing shareholder wealth through maximisation of cash flow.

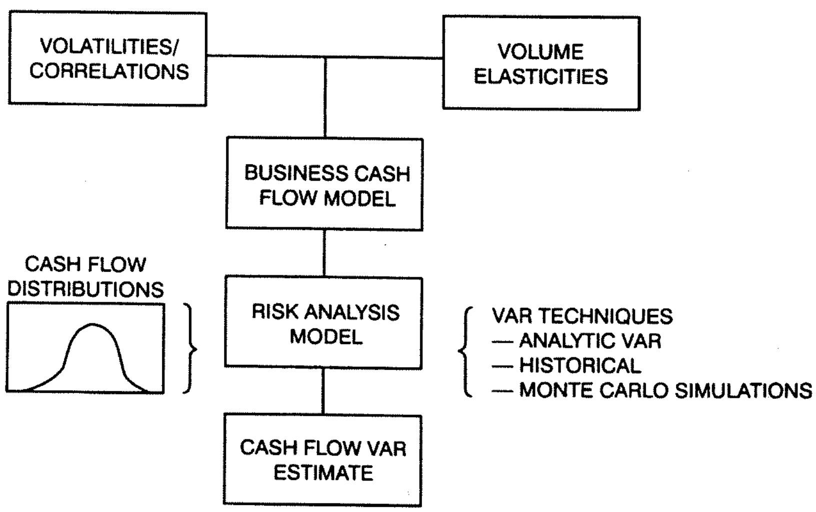
A variation on this approach is to measure the impact of market price changes *on earnings* (a form of earnings VAR).

The overall approach to cash flow VAR is set out in *Exhibit 16.24*.

29. The concept of cash flow or earnings VAR has been suggested by a number of writers; for example, see Chris Turner, "VAR as an Industrial Tool" (1996) 9(3) *RISK* 38-40; David Shinks, "VAR for Corporate" (1996) 9(6) *RISK* 28-29; Leslie McNew, "So Near, So VAR" (1996) 9(10) *RISK* 54-56; see also Karen Spinner, "Companies Put Their Own Spin on VAR" (1996) (Aug) *Global Finance* 48-53; Karen Spinner, "Adapting Value at Risk" (April 1990) 14-23.

Exhibit 16.24

Cash Flow VAR Model



The essential steps in implementing cash flow VAR includes the following:

1. *Risk mapping*—this entails analysis of the entity's exposure to identify the key market prices that affect the cash flow of the firm.
2. *Estimating the historical volatilities and correlations of financial market risk factors relevant to the company*—this is identical to the process developed in the context of more general versions of VAR.
3. *Measuring volume elasticities*—this requires estimation of the *volume sensitivities* of individual cash flow items such as sales and input factors and their correlations to financial market prices. This can be modelled based on historical data series or subjectively based on knowledge of market structures and the industrial economics of the factor. This step is optional in that some approaches to Cash Flow VAR estimation ignore it because of its complexity. However, in reality depending on the importance of volume changes it should be incorporated to capture the changes in portfolio risk composition as a result of volume shifts.
4. *Using the financial variable historical volatilities and correlations and the volume elasticities in the business cash flow model*—this entails incorporating the model variables into the basic cash flow model. This cash flow model will typically incorporate the organisation's own estimates for operating and financial variables which allows analysis of the company's exposures. An example of a business cash flow model is set out in *Exhibit 16.25*. The analysis focuses on the principal traded asset classes and their impact on the projected net cash flow of the entity. Other factors which may be included are alternative sources of

risk such as default (credit) risk on commercial contracts and tax rate changes which may impact on the value of tax losses.

5. *Running the risk model*—the risk model is effectively the VAR risk process. It is identical to that used in the examples used above. In practice, because of the nature of the cash flow model, Monte Carlo Simulation is generally used to create distributions of cash flows based on running a large number of price paths consistent with the historical volatilities, correlations and elasticities.
6. *Derivation of the cash flow VAR estimate*—the cash flow VAR estimate is drawn from the net cash flow distribution generated based on a desired confidence level using a percentile cut-off method. This reflects the typically non normal nature of the cash flow distributions.

Exhibit 16.25

Business Cash Flow Model

IMPACT OF CHANGES IN FINANCIAL MARKET VALUE DRIVERS

CASH FLOW COMPONENTS	PROJECTED CASH FLOW	INTEREST RATES	CURRENCY RATES	COMMODITIES	EQUITIES
Sales Volume					
Sales Price					
Revenue					
Input Volumes					
Input Prices					
Cost Of Sales					
Operating Margin					
Administrative Costs					
Operating Cash Flow Before Tax					
Tax					
Operating Cash Flow After Tax					
Capital Expenditure					
Working Capital Investment					
Net Cash Flow					

The basic approach is consistent for all non-financial institutions, irrespective of industry. The specific industry factors are generally reflected in the risk mapping process and in the design of the business cash flow model.

The risk horizon utilised in estimating cash flow VAR is related to the underlying business cycle. It should be related to the entity's planning horizon over which the cash flows are realised and reaction time scale over which corrective action could be implemented. This means that the VAR risk horizon will be longer than for financial institutions and investment managers. It will generally range from 3 months to 1 year depending on the underlying business dynamics.

7.3.3 *Using cash flow VAR in hedging/risk management*

The interpretation and the use of the cash flow VAR estimate derived focuses on:

- the risk capital requirements suggested by the estimate; and
- the implicit hedging framework derived.

The cash flow VAR estimate represents the statistical or model based risk of change in the company's cash flow over the risk horizon at the stated level of confidence. The estimate generated may be interpreted as the amount capital that the firm has to commit to absorb the potential risk of a fall in cash flows.

This can be illustrated with a simple example. Assume a company has expected cash flow of $800 million over the forecast period of 1 year. Based on an analysis of its risk profile, it estimates its cash flow VAR is $540 million. This implies that at a confidence level of 99%, the worst case cash flow of the company is forecast at $260 million. This compares to anticipated debt service (interest and principal) of $220 million and forecast dividends of $75 million. The risk analysis therefore indicates that there is a possibility that the cash flow of the company may not be sufficient to cover its fixed costs. Re-examining the distribution of cash flow outcomes, it may be possible to identify that there is a 4% probability that cash flow will not cover the identified fixed payments.

The company might consider this possibility undesirable from the point of view of its risk and credit profile and may wish to reduce the risk to its solvency. This can be achieved by holding sufficient capital against the risk of cash flow shortfall. This can be done by holding risk capital of $540 million to reduce the risk of cash flow shortfall to the company.

The actual cost of minimisation of this risk is the cost of capital incurred. Assume the costs of equity capital to the firm to be 13.60% pa.[30] Based on these cost of capital, the effective capital cost of the exposure is $73.4 million pa (calculated as the equity cost of capital on the amount of capital).

The risk capital requirement can be directly related to shareholder value. The firm value can be regarded as two streams of cash flow:

30. This is calculated using the CAPM as follows: Cost of equity = Risk Free Rate (7.00% pa) + Beta (1.1) times Market Risk Premium (6.00% pa).

1. A low risk cash flow stream of $260 which assuming it is sustainable can be capitalised at a low cost of capital.
2. A higher risk cash flow stream of $540 which must be capitalised at a higher cost of capital.

The low risk cash flow stream can be debt financed. The capital held is against the solvency risk presented by the second cash flow stream. In order to minimise the risk of financial distress risk capital must be held against the risk of a cash flow shortfall. This capital can be generated from a number of internal sources (cash reserves, sale of assets, adjustments in capital expenditure etc) or external sources (raising of equity capital). Irrespective of source, the risk capital required to be held will be diverted from other higher returning activity reducing shareholder value; for example, by reducing financial leverage or reducing the capacity to make business investments for a fixed amount of capital.

The analysis of risk can be extended to derive an implicit hedging framework. Hedging action can be considered value creating where the cost of the hedge is lower than the return that can be generated from the released capital. In contrast, hedging activity should not be undertaken where the cost of the hedge exceeds the returns available from the released capital.

In the above example, assume that the underlying price risks can be hedged by a series of transactions. This has the impact of reducing the cash flow VAR to $125 million; a worst case expected cash flow of $675 million. This is considered an acceptable risk level as the fixed payments are effectively adequately covered. The net effect of the hedge is to release approximately $415 million of risk capital which reduces the cost of capital held against cash flow risk by $56.4 million. The hedging action should be undertaken as creating shareholder value where the cost of the hedges is *less than $56.4 million*.

In the above example, the amount at risk was sought to be completely neutralised. In practice, capital allocation or hedging actions designed to adjust risk to within acceptable boundaries consistent with investor preferences.

The use of cash flow VAR in hedging can allow improvement in the efficiency of the hedge. As with financial institutions, the use of cash flow VAR implies the use of VAR reduction as the basis for optimising hedging behaviour through the reduction of risk by entry into transactions which on a portfolio basis reduces the overall cash flow VAR. This type of hedging (referred to as correlation based hedges) can overcome deficiencies in the hedge market (for example, because of liquidity, institutional, or regulatory considerations) as well as reducing the transaction costs of the hedge. This contrasts with specific hedges which would be designed to offset *individual* risk positions which may have lower efficiency or higher costs in effecting an equivalent reduction in risk.

Recent interest in integrated risk management products such as products incorporating correlations or non-linear payoffs can be traced directly to these emerging approaches to the analysis and treatment of risk. An example of this type of product is an interest rate hedge, such as a cap or interest rate swap, which is triggered or activated as a result of an adverse change in a separate defined variable, say a fall in a commodity prices. The structure, a defined exercise option, makes use of the correlation between the interest rates and the commodity price to create a hedge structure which is more cost effective. Central to the structure is the underlying desire to manage the hedger's *cash flow*. An increase in cash flow for the company in this case may be able to be absorbed where it is allied to commodity prices above a specified level. The hedge is designed to operate where a fall in commodity prices *combined* with rising interest rates adversely affects the cash flow of the company outside specified risk tolerances. Similarly, the increased interest in power or exponential options (which pay an exponent of the normal intrinsic value of a standard option) reflect increased understanding of the non-linear nature of certain risks. For example, the impact of demand or supply effects of movements in, say, currency rates on the *volume* of sales or purchases creates specific difficulties in hedging. This is reflective of the assumption underlying hedging generally which is predicated on both known cash flows and linear changes in cash flows.[31]

In essence, the use of Cash Flow VAR to develop an integrated framework for firmwide risk management allows informed and efficient decision-making on:

- Establishing the desired risk profile of businesses.
- Allocating risk capital as between competing business activities based on risk adjusted returns.
- Frameworks for trading and hedging activities to manage financial risk in non-financial corporations.

7.3.4 Market risk management in non-financial institutions—case study

An example of the application of VAR techniques in the management of risk in a non-financial institution is set out in *Appendix B* to this Chapter.

8. RISK ADJUSTED PERFORMANCE MEASUREMENT

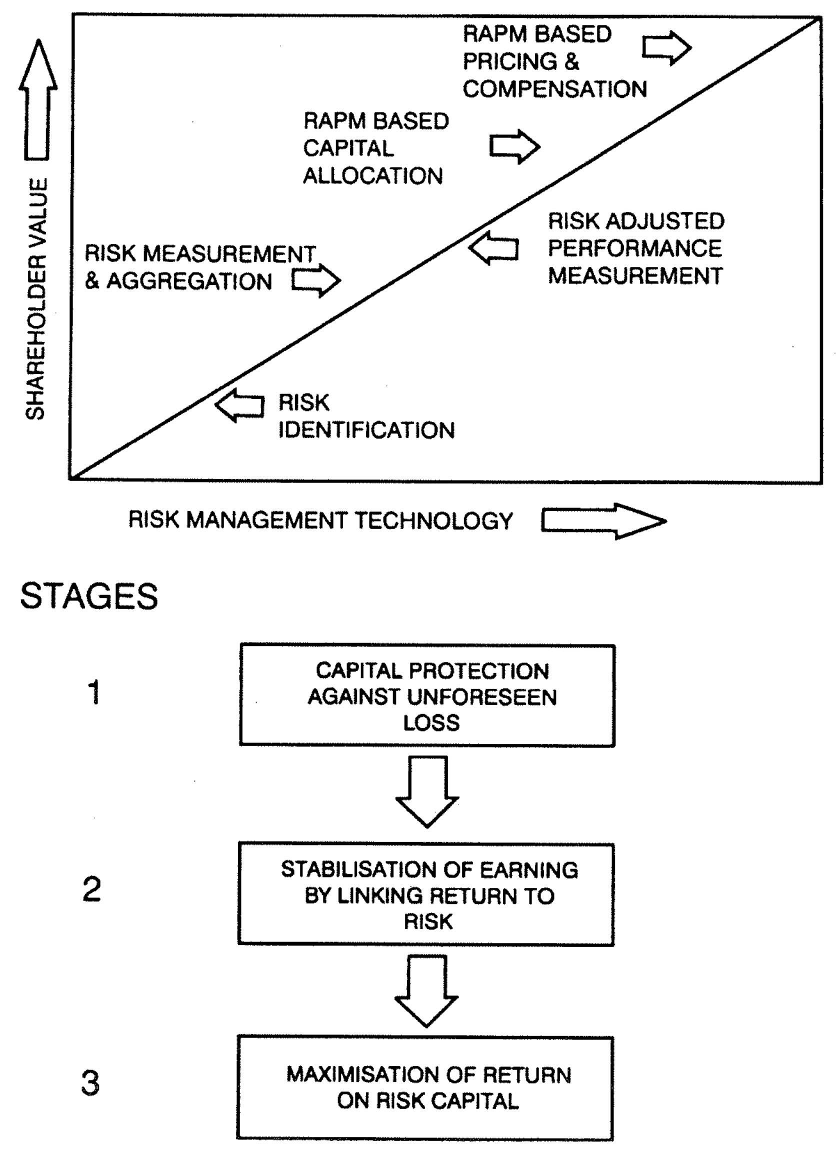
8.1 Overview

The accurate quantification of risk is essential to establishing a link between the returns earned from an activity or business and the risks of that activity or business. The evolution of risk management into risk adjusted performance measurement is set out in *Exhibit 16.26*.

31. See Satyajit Das, *Exotic Options* (IFR Publishing/LBC Information Services, London/Sydney, 1996).

Exhibit 16.26

Evolution of Risk Management



The creation of this link between risk and return entails the implementation of risk adjusted performance measurement (RAPM) systems. Risk adjusted return on capital (RAROC) is one of a number of commonly utilised RAPM systems.

The central element of these systems is the calculation of *risk adjusted returns* of transactions, trading units and businesses. The main advantage of RAPM systems include the ability to facilitate comparability of different businesses enabling capital and resource allocation to be undertaken on a consistent and economically efficient basis consistent with maximisation of shareholder/enterprise value. While primarily developed for financial institutions it can be utilised with some adaptations, as set out above in the discussion of cash flow VAR, for non-financial institutions.

8.2 Concept

Traditional approaches to performance measurement have emphasised revenue or earnings. Typical measures include: gross or net revenue; earning before interest and tax; or net profit.

The major deficiencies of these measures include:

- The earnings are not adjusted for the riskiness of the revenue/earnings stream.
- The earnings are not related to the level of capital required to support the revenue/earning generation activity

RAROC and other RAPM systems overcome these weaknesses in traditional performance measurement techniques by both adjusting earnings for risk and relating it to the level of economic or risk capital committed to the activity. The use of these systems are central to measuring the link between trading activities and shareholder value generation.

8.3 Measurement techniques

The types of measurement techniques utilised can be classified into two separate categories:

- *Generalised measures of risk-return performance*—for example, Sharpe Ratio; Generalised Sharpe Ratio.
- *Risk adjusted return measure*—for example, RORAC models.

The central concept of the Sharpe Ratio is to measure excess return relative to volatility of portfolio value or portfolio returns. Sharpe Ratios are commonly utilised in investment management to measure portfolio performance. The concept of the Generalised Sharpe Ratio is that it relates return from the relevant transaction or activity to the amount of risk measured over a nominated period. This is analogous to the measure of risk adjusted return on the amount of risk capital required. *Exhibit 16.27* sets out details of the calculation of Sharpe Ratios.

Exhibit 16.27

Sharpe Ratios

The Sharpe Ratio is usually defined as:

$$S = (\upsilon - r) / \sigma$$

Where
S = Sharpe Ratio
υ = % return on risky asset portfolio or asset
r = risk free rate
σ = standard deviation of the risky asset or portfolio

As is evident, the Sharpe Ratio determines the return *in excess of the risk free return* relative to the risk of the earnings stream.

The Sharpe Ratio can be generalised to a more broad based measure for any risky activity.

The Generalised Sharpe Ratio is usually defined as:

$$S' = (\upsilon - R) / \sigma \sqrt{T}$$

Where
S' = Generalised Sharpe Ratio
υ = total return on risky asset portfolio or asset over period t (expressed in $)
R = E.r = expected risk free return over time t (expressed in $)
E = equity value invested in asset or activity
r = risk free rate
σ = standard deviation of the risky asset or portfolio
T= number of time periods in time t

Risk adjusted return measures relate to the identification of the economic contribution of an activity through setting income generated by the operation against the shareholder capital required to support its underlying risk. There are a number of risk adjusted capital return measures. The common elements of these measures include:

- adjusting gross earning for the riskiness of the earnings stream;
- determining capital required to be committed to support the activity; and
- calculating the return on the capital base.

The choice of measurement technique focuses on the following:

- RAROC (risk adjusted return on capital);
- RORAC (return on risk adjusted capital); and
- ROVAR (return on value at risk).

The differences between the systems focuses on:

- Whether the adjustment for risk is made by adjusting earnings or the capital held to support the activity.
- The methodology for determining the capital amount which is held.

The generalised RAROC formula is as follows:

Earnings/Economic Risk Capital

Where

Earnings = Revenue − Expenses − Expected Losses ± Adjustments

Economic Capital = Capital required to cover all risks of operation

The individual components of earning are calculated as follows:

- *Revenue* includes all income such as bid-offer spread, fees, commissions, trading income, costs of hedging and funding costs.
- *Expenses* include all direct expenses (salaries, brokerage, premises, technology etc) and allocated costs/overheads.
- *Expected losses* include provisions for expected credit losses, *future* hedging, funding and, operating costs, et cetera.
- *Adjustments* includes any required amendment such as differential tax rates or subsidies.

The individual components of economic risk capital are determined as follows:

- Capital required to cover credit risk (for example, that required under BIS Capital Accord).
- Capital required to cover market risk (for example, the average daily VAR of the activity).
- Capital invested in infrastructure is included as either part of the capital base or treated as an allocated cost (on an amortised basis), which is charged against earnings.

The calculated return allows measurement of the earnings adjusted for risk as a percentage of capital employed.

This return is then compared to a hurdle rate of return required by the entity. This hurdle or benchmark rate can be calculated in several ways:

- The cost of capital of the entity (akin to a divisional cost of capital).
- The expected return on a risk free or risky asset where the quantum is related to the amount of that asset that could be held consistent with risk capital allocated to the relevant business.[32]

The return calculation allows a systematic dissection of performance at several levels:

- The return on economic risk capital earned in absolute terms.
- The capacity on an activity to accrue risk adjusted return on capital at a level adequate to meet the cost of that risk capital.
- The ranking (in relative performance terms) of individual activities.

32. For example, in the equity business it would be feasible to use the VAR or indicative risk limit to work back to an equivalent risk position in *the equity market as a passive investment*. The expected return based on this position calculated as the expected return on equities (based on a risk margin over an appropriate risk-free rate) on the amount of investment. This would then form the benchmark return that would have to be met by the business activities.

8.4 Implementation issues

The key implementation issues relate to development of operational systems to dissect earnings and capital allocation consistent with the model proposed. This requires detailed analysis of:

- earnings sources and attribution to individual activities; and
- costs and appropriate allocation systems.

The key organisational issues include:

- ensuring high level commitment to the concept;
- willingness to implement the system across business boundaries; and
- encouraging the adjustment of pricing and capital utilisation behaviour based on the analysis undertaken.

The implementation of the RAPM system will typically have a number of incidental benefits, including:

- improvements in range and quality of financial performance data;
- improvements in systems;
- improvements in understanding of the earnings dynamics of businesses; and
- enhancement of the overall business decision making framework.

8.5 Applications

The principal applications of RAPM techniques include:

- *Return measurement*—the determination of the relative return of any activity. Return can be measured at multiple levels—individual trader/dealer, desk, business unit, economic entity, et cetera.
- *Evaluation of activity*—the process of return measurement facilitates the rational economic analysis of any activity within a shareholder value framework.
- *Capital management*—utilisation of the evaluation process:
    - to determine capital required to support business activities; and
    - to allocate capital based on returns generated on capital to facilitate growth/expansion of existing businesses.
- Performance linked compensation strategies—using risk adjusted return on capital measures as means for establishing accurate mechanisms for aligning employee performance and compensation with shareholder returns.[33]

33. See Mark Rodrigues, "Compensation Methods in Investment Banks: Tackling the Risk Factor" (1996) (June) *Financial Derivatives and Risk Management* 44-48.

9. SUMMARY

The altered business environment is characterised by an increased reliance on trading and acting as a principal to transactions in financial institutions and increased exposure to financial risk in non-financial institutions. This change is reflected in an increase in market risk exposures of organisations. This shift in risk profile requires a corresponding increase in risk management focus and the development of risk quantification techniques. VAR techniques have emerged as the principal mechanism for the unified quantification of market risk. VAR is a generic term which covers a variety of similar approaches to risk measurement. These techniques allow measurement of market risk which, in turn, can be linked to the management and control of risk, performance evaluation and the allocation of risk capital to activities and businesses.

Appendix A

Market Risk Management in a Financial Institution

The example provided in this Appendix is a simplified outline of some of the primary risk management issues of a financial institution. This covers both the interest rate risk created by its underlying business and the implications of hedging these exposures with interest rate derivatives.

1. OVERVIEW

ABC Bank is a financial institution focusing on retail lending products such as home and personal loans. It provides these loans on a standardised basis and views them as a commodity product which represent easily marketable securities (that is, they can be sold into securitisation vehicles or to other financial institutions). Its lending product mix currently includes both variable and fixed interest products with a term of up to three years. All of its assets and liabilities are denominated in local currency. ABC does not undertake any proprietary trading and it uses derivatives primarily to hedge its underlying balance sheet risks.[i]

Historically, ABC has managed its interest rate risk using "gap" modeling and wishes to move to a value at risk (VAR) methodology. After reviewing the available methodologies it decides to adapt the "Standard Model" from the Bank of International Settlements (BIS) Capital Accord Market Risk Guidelines[ii] (hereafter referred to as BIS standard model). While the underlying methodology is based on the BIS standard model the volatilities and correlations will be adjusted to match the interest rate risk management requirements of ABC (for example, a longer holding period and volatilities and correlations which reflect recent history).

The BIS standard model represents a form of Analytic VAR which uses a standard set of historical volatilities and correlations (often referred to as "Parametric VAR"). Using this methodology ABC Bank aims to achieve the following:

- calculate the bank's current VAR;
- determine appropriate VAR limits to assist in the implementation of hedging strategies;
- use the VAR methodology to fix existing "risk holes";
- incorporate the non-linear pay-off structure of options; and
- introduce the concept of risk adjusted performance measurement.

i. This case study will not distinguish between "traded" and "non-traded" VAR; the focus is on estimating the overall interest rate risk of the group. This creates some difficulty in assessing performance as not all of the bank's underlying assets can be marked-to-market.

ii. See Bank for International Settlements, "Planned Supplement to the Capital Accord to Incorporate Market Risks", April 1995.

2. CURRENT UNDERLYING BALANCE SHEET EXPOSURE

ABC Bank's underlying business creates a naturally ''long'' interest rate bias as it funds all of its activities through largely short-term liabilities and then offers a range of variable and floating rate products. The current mismatch in the balance sheet prior to incorporating derivative positions is set out in *Exhibit 16.28.*

Exhibit 16.28

ABC Bank Gap Report—Underlying Balance Sheet (Excludes Derivatives)

Interest rate gap analysis as at end **Dec 96**

All amounts in $ millions. Quarterly Asset and Liability Repricing

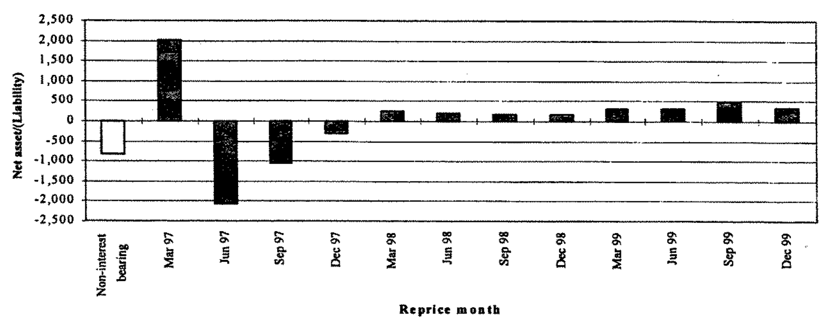
	Non-interest bearing	Mar 97	Jun 97	Sep 97	Dec 97	Mar 98	Jun 98	Sep 98	Dec 98	Mar 99	Jun 99	Sep 99	Dec 99	Total
Assets														
Non-interest assets	574	0	0	0	0	0	0	0	0	0	0	0	0	574
Retail Loans	0	8,419	329	472	764	285	246	242	211	334	317	283	328	12,233
Overdraft Accounts	0	340	0	0	0	0	0	0	0	0	0	0	0	340
Liquidity Assets	0	1,036	286	0	0	0	0	0	0	0	0	193	0	1,515
Total	574	9,795	615	472	764	285	246	242	211	334	317	476	328	14,662
														0
Liabilities														0
Non-interest liabilities	231	0	0	0	0	0	0	0	0	0	0	0	0	231
Savings accounts	0	4,231	0	0	0	0	0	0	0	0	0	0	0	4,231
Investment accounts	0	2,353	2,482	1,515	1,082	25	34	60	42	10	0	0	0	7,603
Wholesale Borrowings	0	1,197	220	0	0	0	0	0	0	0	0	0	0	1,417
Capital Market Issues	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Equity	1,180	0	0	0	0	0	0	0	0	0	0	0	0	1,180
Total	1,411	7,781	2,702	1,515	1,082	25	34	60	42	10	0	0	0	14,662
Net asset/(liability)	(836)	2,014	(2,087)	(1,043)	(318)	260	212	182	169	324	317	476	328	0

Currently ABC only offers fixed rate loans with a fixed interest rate term of one, two or three years.

The gap report highlights the "long" fixed rate interest rate position created by the funding of fixed rate assets out to a three year term with primarily floating rate liabilities (this does not include hedge derivatives). This is illustrated in *Exhibit 16.29.*

Exhibit 16.29

ABC Bank—Quarterly Gap Graph



3. MODIFIED EXPOSURE—THE IMPACT OF DERIVATIVES

ABC has traditionally hedged fixed rate loans on an ongoing basis. Previous policy has been to hedge between 50% and 90% of fixed rate loans with an interest as they are made. Up until the present all exposures have been hedged with paying fixed receiving floating interest rate swaps. The modified exposure balance sheet after allowing for the current derivatives portfolio is set out in *Exhibit 16.30.*

Exhibit 16.30

ABC Bank Gap Report—Including Derivative Hedge Positions

Interest rate gap analysis as at end **Dec 96**

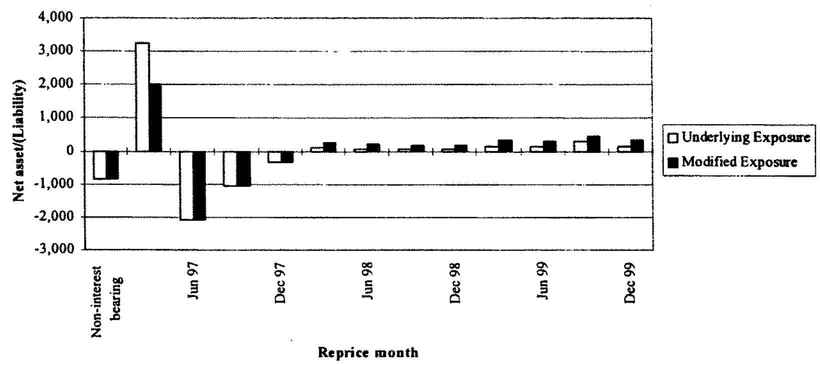
All amounts in $ millions. Quarterly Asset and Liability Repricing

	Non-interest bearing	Mar 97	Jun 97	Sep 97	Dec 97	Mar 98	Jun 98	Sep 98	Dec 98	Mar 99	Jun 99	Sep 99	Dec 99	Total
Assets														
Non-interest assets	574	0	0	0	0	0	0	0	0	0	0	0	0	574
Retail Loans	0	8,419	329	472	764	285	246	242	211	334	317	283	328	12,233
Overdraft Accounts	0	340	0	0	0	0	0	0	0	0	0	0	0	340
Liquidity Assets	0	1,036	286	0	0	0	0	0	0	0	0	193	0	1,515
Total	574	9,795	615	472	764	285	246	242	211	334	317	476	328	14,662
														0
Liabilities														0
Non-interest liabilities	231	0	0	0	0	0	0	0	0	0	0	0	0	231
Savings accounts	0	4,231	0	0	0	0	0	0	0	0	0	0	0	4,231
Investment accounts	0	2,353	2,482	1,515	1,082	25	34	60	42	10	0	0	0	7,603
Wholesale Borrowings	0	1,197	220	0	0	0	0	0	0	0	0	0	0	1,417
Capital Market Issues	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Equity	1,180	0	0	0	0	0	0	0	0	0	0	0	0	1,180
Total	1,411	7,781	2,702	1,515	1,082	25	34	60	42	10	0	0	0	14,662
Derivatives		1,333	0	0	(11)	(164)	(134)	(115)	(107)	(207)	(203)	(181)	(210)	0
Net asset/(liability)	(836)	3,347	(2,087)	(1,043)	(329)	96	79	67	62	117	114	295	118	0

The current hedging policy has reduced the "long" interest rate exposure by converting a large portion of the fixed rate loan portfolio to floating rate. This reduced sensitivity to interest rate movements is reflected in the gap graph provided in *Exhibit 16.31.*

Exhibit 16.31

ABC Bank—Quarterly Gap Graph



4. CALCULATING VALUE AT RISK

Using the modified balance sheet position set out in *Exhibit 16.30* as the starting point we can calculate the VAR by applying the BIS Standard model. The steps in this process are as follows:

1. *Quantify exposure*: Divide the interest rate risks into the ten time bands or "buckets" used by the BIS standard model.
2. *Determine risk equivalents*: Determine the "risk equivalent" position in each time bucket by multiplying the assets and liabilities in each time bucket by the interest rate volatility and assumed duration of each time bucket.
3. *Calculate outright risk*: The outright exposure to interest rates is given by summing across the risk equivalent in each time bucket.
4. *Calculate yield curve risk*: Determine the amount of risk equivalents which offset in the calculation of the outright risk in step 3 (referred to as "horizontal risk") and include an amount for the potential for loss arising from a change in the shape of the yield curve rather than a general rise or fall in rates.
5. *Calculate Basis Risk*: Within each time bucket, determine those assets and liabilities which are offsetting and include an amount for the possibility of loss arising from the fact that the interest rate risk on these assets and liabilities is not perfectly correlated (referred to as "vertical risk").

6. *Calculate VAR*: The VAR is given by summing the outright risk, yield curve risk and basis risk.[iii]

Each of these steps is set out in the following subsections.

4.1 Divide exposures into time bands

The BIS standard methodology assumes that the yield curve can be divided into three "zones" which are then divided into ten "time bands". ABC's current balance sheet structure divides into the following time buckets (the amount repricing in each time bucket reflects the market value of each asset and liability):

iii. All risks are expressed as absolute values.

Exhibit 16.32

Principal Maturity Profile by Time Bucket and Zone

All amounts in millions

Time Bucket	Non-interest bearing	Zone 1			Zone 2			Zone 3				Total
		0-3m	3-6m	6-12m	1-2yr	2-3yr	3-4yr	4-5yr	5-7yr	7-10yr	>10yr	
Assets												
Non-interest assets	574											574
Retail Loans		8,585	329	1,209	939	1,171						12,233
Overdraft Accounts		340	0	0	0	0						340
Liquidity Assets		1,036	286	0	0	193						1,515
Total	574	9,962	615	1,209	939	1,364	0	0	0	0	0	14,662
Liabilities												
Non-interest liabilities	231											231
Savings accounts		4,231	0	0	0	0						4,231
Investment accounts		2,353	2,482	2,597	161	10						7,603
Wholesale Borrowings		1,197	220	0	0	0						1,417
Capital Market Issues		0	0	0	0	0						0
												0
Total	231	7,781	2,702	2,597	161	10	0	0	0	0	0	13,483
Derivatives		1,333	0	(11)	(520)	(771)						
(-ve net liab)												
Net Asset/Liability	343	3,513	(2,087)	(1,400)	259	582	0	0	0	0	0	
Equity	1,180											
Net A/L by Zone				26			841				0	

Exhibit 16.32—*continued*

Principal Maturity Profile by Time Bucket and Zone

All amounts in millions

Time Bucket	Non-interest bearing	Zone 1			Zone 2			Zone 3				Total
		0-3m	3-6m	6-12m	1-2yr	2-3yr	3-4yr	4-5yr	5-7yr	7-10yr	>10yr	
Assets												
Non-interest assets	574,178											574,178
Retail Loans		8,585,208	328,724	1,208,856	939,156	1,170,562				0		12,232,506
Overdraft Accounts		340,186	0	0	0	0				0		340,186
Liquidity Assets		1,036,200	286,000	0	0	193,160				0		1,515,360
Total	574,178	9,961,594	614,724	1,208,856	939,156	1,363,722	0	0	0	0	0	14,662,230
Liabilities												
Non-interest liabilities	231,000											231,000
Savings accounts		4,230,732	0	0	0	0				0		4,230,732
Investment accounts		2,353,120	2,481,732	2,597,122	161,128	10,362				0		7,603,464
Wholesale Borrowings		1,197,416	220,066	0	0	0				0		1,417,482
Capital Market Issues		0	0	0	0	0				0		0
												0
Total	231,000	7,781,268	2,701,798	2,597,122	161,128	10,362	0	0	0	0	0	13,482,678
Derivatives		1,332,879	0	(11,476)	(519,519)	(770,908)				0		
(-ve net liab)												
Net Asset/Liability	343,178	3,513,205	(2,087,074)	(1,399,742)	258,509	582,452	0	0	0	0	0	
Equity	1,179,552											
Net A/L by Zone				26,389			840,961				0	

Given the range of ABC's activities, all of the interest rate exposures fall into the first five time buckets.

The fixed term asset amounts in this table differ from the gap report as an adjustment has been made for the expected level of prepayment on the fixed rate assets.

4.2 Determine risk equivalents

Risk equivalents are calculated by converting the face value amounts from section 4.1 into an amount which represents the "worst case" change in value of the assets and liabilities in each time bucket. Under the BIS methodology this is done as follows:

$$\text{Risk Equivalent} = \text{Market Value} \times \text{Volatility} \times \text{Duration}$$

Where

Market Value = The Market value of each asset and liability in the time bucket.

Volatility = A Historical interest rate 99% confidence volatility for the time horizon used.

Duration = The assumed duration of each time bucket

Risk Weight = Volatility × Duration

The BIS standard model assumes the combination of volatilities and durations set out in *Exhibit 16.33.*

Exhibit 16.33

BIS Standard Model Risk Weights

Zone	Zone 1			Zone 2			Zone 3			
Bucket	0-3m	3-6m	6-12m	1-2y	2-3y	3-4y	4-5y	5-7y	7-10y	>10y
Volatility(% pa)	1.00%	1.00%	1.00%	0.90%	0.80%	0.75%	0.75%	0.70%	0.65%	0.60%
Duration	0.20	0.40	0.70	1.39	2.19	3.00	3.67	4.64	5.77	7.50
Risk Weight	0.20%	0.40%	0.70%	1.25%	1.75%	2.25%	2.75%	3.25%	3.75%	4.50%

However, the BIS standard model has been developed for a bank trading portfolio which consists of liquid financial instruments held for trading purposes and the assumed VAR time horizon is ten days. Given ABC's underlying assets are less liquid and the time taken to implement risk management strategies may be several weeks then a time horizon of one month is more appropriate to ABC's requirements. Further, ABC would like the volatility factors to reflect recent interest rate history[iv] and also that durations should reflect the current level of interest rates and average term of each time bucket.

Accordingly, ABC has calculated its own 99% confidence risk weights as set out in *Exhibit 16.34*.[v]

iv. A feature of interest rate risk volatilities in the 1990s has been greater volatility in the medium term time buckets than the shorter term buckets—this is the inverse of the volatility hue profile assumed in the BIS risk weights.

v. Given ABC Bank operates in an OECD country and if it was applying the BIS methodology to a trading portfolio then the VAR would have to be calculated according to the BIS risk weights.

Exhibit 16.34

ABC Bank Modified Risk Weights

Zone	Zone 1			Zone 2			Zone 3			
Bucket	0-3m	3-6m	6-12m	1-2y	2-3y	3-4y	4-5y	5-7y	7-10y	>10y
Volatility(% pa)	0.62%	0.75%	0.67%	0.45%	1.08%	1.06%	1.03%	1.0%	0.96%	0.96%
Duration	0.20	0.40	0.70	1.39	2.19	3.00	3.67	4.64	5.77	7.50
Risk Weight	0.12%	0.30%	0.61%	1.35%	2.36%	3.17%	3.78%	4.64%	5.54%	7.20%

Applying these risk weights to the market values from section 4.2 gives the risk equivalents set out in *Exhibit 16.35.*

Exhibit 16.35

Risk Equivalent Maturity Profile by Time Bucket and Zone

All amounts in $ millions

Time Bucket	Non-interest bearing	Zone 1			Zone 2			Zone 3				Total
		0-3m	3-6m	6-12m	1-2yr	2-3yr	3-4yr	4-5yr	5-7yr	7-10yr	>10yr	
Risk Weight		0.12%	0.30%	0.61%	1.35%	2.36%	3.17%	3.78%	4.64%	5.54%	7.20%	
Assets												
Retail Loans		10.65	0.98	7.36	12.72	27.65						59.36
Overdraft Accounts		0.42	0.00	0.00	0.00	0.00						0.42
Liquidity Assets		1.28	0.85	0.00	0.00	4.56						6.70
Total	0.00	12.35	1.83	7.36	12.72	32.22	0.00	0.00	0.00	0.00	0.00	66.48
Liabilities												
Non-interest liabilities	0.00											0.00
Savings accounts		5.25	0.00	0.00	0.00	0.00						5.25
Investment accounts		2.92	7.40	15.82	2.18	0.24						28.56
Wholesale Borrowings		1.48	0.66	0.00	0.00	0.00						2.14
Capital Market Issues		0.00	0.00	0.00	0.00	0.00						0.00
												0.00
Total	0.00	9.65	8.05	15.82	2.18	0.24	0.00	0.00	0.00	0.00	0.00	35.94
Derivatives		1.65	0.00	(0.07)	(7.04)	(18.21)						(23.66)
(-ve net liab)												
Net Asset/Liability	0.00	4.36	(6.22)	(8.52)	3.50	13.76	0.00	0.00	0.00	0.00	0.00	6.87
Equity	0.00											
Net A/L by Zone				(10.39)			17.26				0.00	

4.3 Outright value at risk

The outright value at risk is ABC's exposure to a parallel shift in the yield curve. It is given by the summation of the risk equivalents across all time buckets. Using the data from *Exhibit 16.34* the outright risk calculation is as follows:

Time Bucket	Risk Equivalents ($m)
0-3m	4.356
3-6m	−6.219
6-12m	−8.524
1-2yr	3.501
2-3yr	13.760
Outright Risk	6.874

ABC has a net "long" interest rate exposure of $6.874 million. If interest rates increase across the yield curve by the amounts assumed in the volatilities determined in section 4.2, the estimated present value of the loss is $6.874 million.

4.4 Calculate yield curve risk

Yield curve risk (or "horizontal risk") represents the exposure to risk from a change in the shape of the yield curve. The outright risk assumes perfect correlation in interest rates across all time buckets, the yield curve risk calculation looks at the net risk equivalent in each zone and time bucket and identifies those time buckets which have been offset (this amount is referred to as the "horizontal disallowance"). Once these offsets have been identified, a risk amount is calculated which reflects the extent to which the offsetting time buckets are not correlated.

The BIS model assumes fixed parameters for the correlation between time buckets and zones. ABC has recalculated this correlation based on actual historical behaviour of its assets and liabilities. Both correlation tables are set out in *Exhibit 16.36.* Once these correlations have been determined then the disallowance or offset rates are given by the difference between 100% and the correlation. Two forms of disallowance are calculated—"intrazone" and "interzone". Intrazone disallowances relate to offsetting risk amounts within a zone while interzone risks relate to offsetting amounts across zones.

Exhibit 16.36

BIS Standard Model Correlations

Assumed Correlation

	Zone 1	Zone 2	Zone 3
Zone 1	60%		
Zone 2	60%	70%	
Zone 3	0%	60%	70%

Resulting Disallowance

	Zone 1	Zone 2	Zone 3
Zone 1	40%		
Zone 2	40%	30%	
Zone 3	100%	40%	30%

ABC Modified Correlations

Assumed Correlation

	Zone 1	Zone 2	Zone 3
Zone 1	70%		
Zone 2	50%	80%	
Zone 3	30%	70%	80%

Resulting Disallowance

	Zone 1	Zone 2	Zone 3
Zone 1	30%		
Zone 2	50%	20%	
Zone 3	70%	30%	20%

The correlation matrices in *Exhibit 16.35* show both the intrazone and interzone disallowances. With the interzone correlations on the diagonal matrix row (for example, the zone 1 intrazone disallowance is given by the zone 1: zone 1 disallowance).

The table in *Exhibit 16.37* sets out the calculation of offsetting risk amounts across zones and time buckets (using the ABC yield curve correlations). The current yield curve risk is $6.501 million—that is, the present value of the loss arising from an unfavourable movement in the yield curve (for example, in ABC's case, a positive steepening in the yield curve).

Exhibit 16.37
Horizontal Disallowance

(Expressed as risk equivalent positions, $ m)

	Zone 1			Zone 2			Zone 3		
	0-3m May-97	3-6m Aug-97	6-12m Dec-97	1-2yr Sep-98	2-3yr Sep-99	3-4yr Sep-00	4-5yr Sep-01	5-7yr Mar-03	7-10yr Sep-05
Net Asset by Bucket	4.356	— 6.219	— 8.524	3.501	13.760	—	—	—	—
Net Asset by Zone			— 10.388			17.261			

Offset Risk Amounts $'000's

	Zone 1	Zone 2	Zone 3
Zone 1	4.356		
Zone 2	10.388	—	
Zone 3	—	—	—

Disallowance Rates

	Zone 1	Zone 2	Zone 3
Zone 1	30%		
Zone 2	50%	20%	
Zone 3	50%	30%	20%

Horizontal Disallowance Amounts $'000's

	Zone 1	Zone 2	Zone 3
Zone 1	1.307		
Zone 2	5.194	—	
Zone 3	—	—	—

Summary VAR	
Intrazone	1.307
Interzone	5.194
Total	6.501

4.5 Calculate basis risk

Basis risk arises from the fact that the yields of different assets and liabilities in the same time bucket are not perfectly correlated (referred to as ''vertical risk''). As with the other risks areas, ABC has modified the basis risk correlations from the original BIS standard model. The revised basis risk, based on ABC's revised correlations, is set out in *Exhibit 16.38.*

Exhibit 16.38

Vertical Disallowance Calculations

	Total	Zone 1			Zone 2	
		0-3m	3-6m	6-12m	1-2y	2-3y
Offset Risk ($m)	25.644	0.200	0.196	—	7.035	18.213
Vertical Disallowance Rate	—	30%	15%	15%	15%	15%
Vertical Disallowance ($m)	3.877	0.060	0.029	—	1.055	2.732

The estimated present value of losses arising from less than perfectly correlated movements between assets and liabilities is \$3.877m.

4.6 Calculate total VAR

The total VAR is given by summing all three elements of risk. This is illustrated in *Exhibit 16.39*. The total VAR of \$17.251m represents 1.22% of capital and 5.89% of net interest income (NII). While appearing relatively low at first, the VAR represents the loss over a monthly time horizon, on an annualised basis the loss is estimated to be 4.24% of capital and 20.4% of NII.[vi]

vi. This annual result has been calculated by multiplying the monthly figure by the square root of 12. As such it can only be treated as an approximation as the 99% distribution of annual data is likely to be less than implied by multiplying by the square root of 12 and it assumes nothing would be done to change the exposure over one year.

Exhibit 16.39
ABC Bank Value at Risk Model

As at = 31-Dec-96

Maturity Profile

Time Zone		Zone 1			Zone 2			Zone 3				
Time Bucket		0-3m	3-6m	6-12m	1-2yr	2-3yr	3-4yr	4-5yr	5-7yr	7-10yr	>10yr	Totals
End of Bucket		Mar-97	Jun-97	Dec-97	Dec-98	Dec-99	Dec-00	Dec-01	Dec-03	Dec-07	Dec-09	
Net Asset/(Liability)	Bucket	3,513	(2,087)	(1,400)	259	582	0	0	0	0	0	867
$m	Zone			26,389			840,961				0	
Risk Equivalents	Bucket	4.356	−6.219	−8.524	3.501	13.760	0.000	0.000	0.000	0.000	0.000	6.874
$m	Zone			−10.388			17.261				0.000	

Value at Risk Assumed Holding Period: 4 weeks

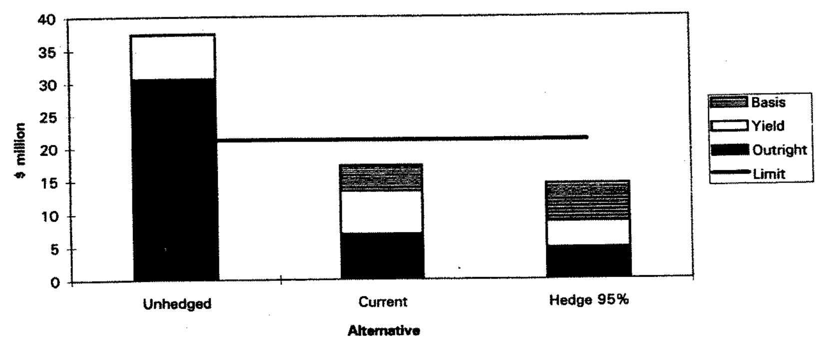
99% confidence level, actual history	**Risk** $m	**Description of Sensitivity**
Outright Exposure to Interest Rates	**6.874**	Exposure to a rise in interest rates
Yield Curve Risk	**6.501**	Exposed to a positive move in the yield curve
Basis Risk	**3.877**	General Asset & Liability mismatch
Total as a present value amount (mthly)	**17.251**	Worst case loss as a present value amount
Total as a present value amount (annual)	**59.759**	

Risk comparatives	mthly	annual
VAR as a % of capital	1.22%	4.24%
VAR as a % of Net Interest Income	5.89%	20.40%

After deriving the model it becomes a powerful tool for analysing the risk sensitivity of ABC's balance sheet. For example, it is useful to examine the risk impact of different hedging strategies and *Exhibit 16.40* calculates the VAR of the underlying balance sheet with no derivative hedges, the current position and hedging 95% of the fixed rate assets with an interest term greater than 1 year.

Exhibit 16.40

VAR and Hedging Alternatives



Not surprisingly the VAR of the unhedged position is the greatest. A little more interesting is the fact that the incremental fall from the current position to 95% hedged is fairly small. This is because while the outright and yield curve risk have declined the basis risk has substantially increased. This reflects one of the inadequacies of the methodology used—a basis risk is assumed to exist between the fixed rate assets and the derivatives used to hedge them. While some basis risk obviously does exist, it is substantially less than the VAR estimates suggest and can be reduced by either reducing the basis risk disallowance or by offsetting between derivatives and fixed rate assets. For the purposes of this example, we will continue to leave the basis risk calculations unchanged but will focus more on the outright and yield curve risks.

4.7 Setting VAR-based risk limits

VAR provides a summary statistic of ABC Bank's current interest rate risk and as such provides a convenient method of limiting the bank's interest rate exposures. A common approach is to set VAR risk limits as a percentage of an organisation's capacity to absorb losses—that is, its capital base or operating earnings.

While an overzealous risk manager may wish to remove all risk from a balance sheet this is inappropriate for most financial institutions for the following reasons:

1. By removing all risk from the balance sheet finance theory tells us that it also reduces the likely return.
2. The "positioning" of the balance sheet exposure to take advantage of expected interest rate movements is an important source of income for most financial institutions.
3. A component of the balance sheet position is funded by the banks capital, by hedging all assets and liabilities to a floating rate risk it assumes that the return on capital should be linked to the floating interest rate plus the net interest margin. This cannot be assumed and given the longer term assets being funded by the bank it is likely the return on these assets should bear some relationship to the expected life of the assets.

With these factors in mind, ABC sets a VAR limit relative to its capital base such that the monthly VAR cannot exceed 1.5% of capital (currently $21.15 million). This is viewed to allow sufficient capital and balance sheet positioning while maintaining a reasonable level of risk for an organisation such as ABC. As the line in *Exhibit 16.40* shows, ABC is currently within the limits however if it reduces its level of hedging then this limit will be exceeded.

4.8 Using VAR to fix "risk holes"

As the gap graph in *Exhibit 16.31* indicated, even with the current hedge portfolio, ABC has a substantial mismatch in the interest rate maturity of assets and liabilities under one year. Currently, a $3.3 billion dollar net asset exists in the March 97 time bucket versus net liabilities in the June and September time buckets of $2.1 billion and $1.0 billion respectively.

The Asset and Liability Committee (ALCO) is concerned that short term interest rates will decline over the next nine months leading to a net erosion of interest margin as the yield on assets declines and liabilities remain fixed.

In order to protect this position ABC elects to hedge the exposure by purchasing interest rate floors with at-the-money strikes in June and September 1997 as follows:

Summary of Floors

Expiry	Mar-97	Sep-97
Face Value ($ m)	1,000	500
Delta	48%	36%
Gamma	1.19	0.93
Vega	1.90	2.23

Incorporation of options into the BIS standard methodology can be done in a number of ways, however a reasonably accurate approach is the "delta plus" approach which uses the delta equivalent of positions and then adds an amount for gamma and vega risk. Given the derivative decomposition rules from section 5.2 of this chapter the delta equivalent of the floors is as follows:

Derivatives Portfolio—Floors "Delta Plus" calculation

Time Bucket	Dec 96	Jun 97	Sep 97
Face Value		1,000,000	500,000
Strike		7.00	7.00
Delta		48%	36%
Gamma		Ignore	Ignore
Vega		1.90	2.23
Delta position ($ m)	−666	484	182
Vega Risk ($m)		0.100	0.063

The primary influence of the floors is to increase create net assets in the second and third time bucket of $666 million and a corresponding liability in the first bucket. In this case because the options have positive gamma no VAR "add-on" is required while a vega add-on of $163,000 is required. The total VAR after the implementation of the floors is set out in *Exhibit 16.41*.

Exhibit 16.41

ABC Bank Value at Risk Model—After Executing Caps

As at = 31-Dec-96

Maturity Profile

Time Zone		Zone 1			Zone 2			Zone 3				
Time Bucket / End of Bucket		0-3m Mar-97	3-6m Jun-97	6-12m Dec-97	1-2yr Dec-98	2-3yr Dec-99	3-4yr Dec-00	4-5yr Dec-01	5-7yr Dec-03	7-10yr Dec-07	>10yr Dec-09	Totals
Net Asset/(Liability)	Bucket	2,182	−1,603	−1,218	259	582	0	0	0	0	0	202
$'000	Zone			26,389			840,961				0	
Risk Equivalents	Bucket	2.705	−4.777	−7.417	3.501	13.760	0.000	0.000	0.000	0.000	0.000	7.772
$'000	Zone			−9.489			17.261				0.000	

Value at Risk

Assumed Holding Period: 4 weeks

99% confidence level, actual history	**Risk** $'000's	**Description of Sensitivity**
Outright Exposure to Interest Rates	**7.772**	Exposure to a rise in interest rates
Yield Curve Risk	**5.556**	Exposed to a positive move in the yield curve
Basis Risk	**4.249**	General Asset & Liability mismatch
Vega Risk	**0.163**	Long Volatility Risk
Total as a present value amount (mthly)	**17.740**	Worst case loss as a present value amount
Total as a present value amount (annual)	**61.453**	

Risk comparatives	mthly	annual
VAR as a % of capital	1.26%	4.36%
VAR as a % of Net Interest Income	6.05%	20.97%

Interestingly, the VAR has increased to \$17.7m from \$17.21m. This is because by reducing some of the net liability in zone 1 ABC has reduced some of the yield curve offsetting. As a result outright risk has increased more than the yield curve risk has decreased. In addition there is the addition of the vega risk associated with a 25% fall in the implied volatility in the floor of \$0.163m.

Appendix B

Market Risk Management in a Resource Company

The example provided in this Appendix is a simplified examination of the calculation of VAR for a corporation—in this case an oil producer. The case study looks to break the oil companies market risk into the three main drivers (the oil price, exchange rate and interest rates) and calculate an estimate of market risk.

1. OVERVIEW

Let us suppose that GIANT Oil Ltd is a limited liability company which produces and explores for crude oil in Asia and Australia. It is an Australian-based company and is listed on a local stock exchange. Expected daily production in 1997 is estimated to be around 10,000 barrels/day rising to around 13,000 barrels/day in 2001.[i]

Traditionally, GIANT has hedged its exposure to the oil price by hedging 30% of the next year's budgeted production. However, it has now decided to manage all of its market risks on a consistent basis and will use VAR to provide a consistent methodology for assessing its expected level of risk.

A summary of GIANT's budgeted balance sheet and income statement for 1997 is set out in *Exhibit 16.42*.

Exhibit 16.42

Budgeted Results for 1997

Assumptions		
Oil Price (US$/Barrel)	$	20.00
FX Rate (A$1 = ?US$)		0.7500
Three Month US$ LIBOR		6.00%

Balance Sheet —Dec 97	A$'000's	
Total Assets		290,000
Borrowings		80,000
Other Liabilities		53,333
Net Assets		156,667

i. This does not include any estimate for production from new oil fields arising from current exploration.

Exhibit 16.42—*continued*

Income Statement	A$'000's
Revenues	99,490
Expenses	
Variable Expenses	11,541
Interest Expenses	3,200
Fixed Expenses	33,333
Total Expenses	48,074
Operating Profit (1)	51,415

(1) After interest before tax and depreciation

All oil produced is sold under contract based off a New York Harbour West Texas Intermediate (WTI) price. Because most receipts are denominated in US$ GIANT has denominated its borrowings in US$—the forecast debt level is US$60m (A$80m) for all of 1997. Currently, the borrowings are floating rate and based off US$ LIBOR.

The bulk of the variable and fixed operating expenses are denominated in US$.

2. CURRENT UNDERLYING EXPOSURE

GIANT has three main sources of market risk:

- the US$ oil price;
- the A$/US$ exchange rate; and
- US$ LIBOR.

The list also reflects the order of importance GIANT currently attributes to each of the risk drivers.

Based on a forecast crude oil price of US$20/BBL, exchange rate of 0.7500 and interest rate of 6%, the expected future operating performance is set out in *Exhibit 16.43*.

Exhibit 16.43

Forecast Production Performance—Next 5 Years

(As at 31 Dec 1996)

Year	Av Price	Barrels p.a	Revenues	Expenses	Profit
Dec-97	26.67	3,730,858	99,489,534	48,074,119	51,415,415
Dec-98	26.67	4,033,671	107,564,566	49,677,490	57,887,077
Dec-99	26.67	4,275,707	114,018,849	51,106,187	62,912,663
Dec-00	26.67	4,532,178	120,858,082	52,593,138	68,264,945
Dec-01	26.67	4,803,901	128,104,037	54,141,140	73,962,896

3. SENSITIVITY TO MARKET RISKS

GIANT carries a substantial exposure to market risk. This is not surprising as a commodity producing company is typically viewed as a risk taking entity and the capital structure and share price usually reflect that structure.

This market risk sensitivity is illustrated in the sensitivity table set out in *Exhibit 16.44*. This shows the sensitivity of GIANT's revenues to movements in both the oil price and exchange rate.

Exhibit 16.44

Revenue Sensitivity — 1997

US$ Oil Price (WTI)

		$ 17.00	$ 18.00	$ 19.00	$ 20.00	$ 21.00	$ 22.00	$ 23.00
	0.72	−11,399,842	− 6,218,096	−1,036,349	4,145,397	9,327,144	14,508,890	19,690,637
Exchange	0.73	−12,606,551	− 7,495,787	−2,385,023	2,725,741	7,836,504	12,947,268	18,058,032
Rate	0.74	−13,780,645	− 8,738,946	−3,697,246	1,344,453	6,386,153	11,427,852	16,469,551
	0.75	−14,923,430	− 9,948,953	−4,974,477	—	4,974,477	9,948,953	14,923,430
	0.76	−16,036,142	−11,127,119	−6,218,096	−1,309,073	3,599,950	8,508,973	13,417,996
	0.77	−17,119,952	−12,274,683	−7,429,413	−2,584,144	2,261,126	7,106,395	11,951,665
	0.78	−18,175,973	−13,392,822	−8,609,671	−3,826,521	956,630	5,739,781	10,522,932

A negative sign indicates a decline in revenues

The sensitivity table for the US$ interest expense (converted to A$) is provided in *Exhibit 16.45*.

Exhibit 16.45
Interest Expense Sensitivity (A$)

US$ LIBOR - %pa

	0	3.00	4.00	5.00	6.00	7.00	8.00	9.00
	0.72	2,500,000	1,666,667	833,333	—	−833,333	−1,666,667	−2,500,000
Exchange	0.73	2,465,753	1,643,836	821,918	—	−821,918	−1,643,836	−2,465,753
Rate	0.74	2,432,432	1,621,622	810,811	—	−810,811	−1,621,622	−2,432,432
	0.75	2,400,000	1,600,000	800,000	—	−800,000	−1,600,000	−2,400,000
	0.76	2,368,421	1,578,947	789,474	—	−789,474	−1,578,947	−2,368,421
	0.77	2,337,662	1,558,442	779,221	—	−779,221	−1,558,442	−2,337,662
	0.78	2,307,692	1,538,462	769,231	—	−769,231	−1,538,462	−2,307,692

A negative sign indicates a rise in expense

4. CALCULATING VALUE AT RISK

The methodology for estimating VAR for a corporation is not as clear cut as for a financial institution. In the case study in Appendix A all financial assets and liabilities (except capital) had a defined interest rate term. In the case of a corporation the term of the market risks is unknown and in this case study they are likely to exist as long as GIANT continues to sell oil.

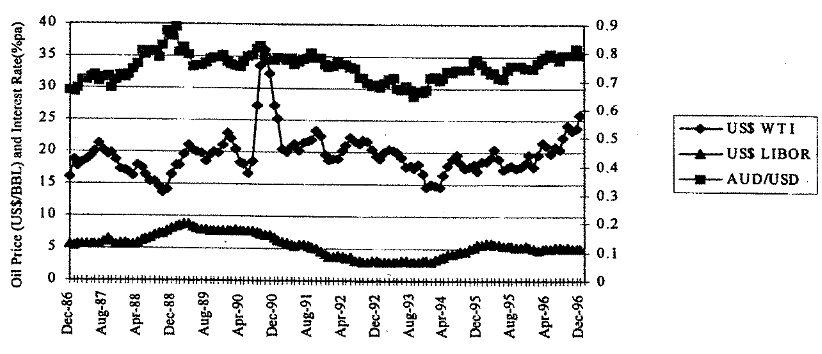
The approach GIANT decides to take in estimating VAR is to determine the likely "worst case" impact of adverse market movements in annual operating profit using a form of the variance/covariance methodology. Accordingly, it will need to estimate the "worst case" movements for each market risk and their correlations, and then apply these to the annual exposures.

4.1 Determine historical volatilities

GIANT takes a long-term view of its market risks and elects to calculate volatilities and correlations based on monthly observations for the past ten years. It will then use this data to calculate the 90% confidence level annual VAR. *Exhibit 16.46* plots the historical oil price, exchange rate and interest rates monthly for the ten years to December 1996.

Exhibit 16.46

Underlying Price Movement



Using this historical data the volatilities and correlations are calculated in *Exhibit 16.47.*

Exhibit 16.47

Summary Volatility and Correlation Coefficients

Monthly	67% Price Volatility	95% Price Volatility	Correlation Coefficient US$ WTI	AUD/USD	US$ LIBOR
US$ WTI	7.56%	12.48%	1.00	0.09	0.08
AUD/USD	2.57%	4.24%	0.09	1.00	0.16
US$ LIBOR	4.03%	6.65%	0.08	0.16	1.00

Annualised	67% Price Volatility	95% Price Volatility	Correlation Coefficient US$ WTI	AUD/USD	US$ LIBOR
US$ WTI	26.20%	43.24%	1.00	0.09	0.08
AUD/USD	8,91%	14.69%	0.09	1.00	0.16
US$ LIBOR	13.97%	23.05%	0.08	0.16	1.00

The analysis assumes that each of these data series is normally distributed. *Exhibit 16.48* compares the actual distribution of each series to a normal distribution. While none of the distributions follow a strictly normal shape as a first estimate they are viewed by GIANT as adequate.

Exhibit 16.48

Frequency Histogram—WTI
Monthly 1987 to 1996

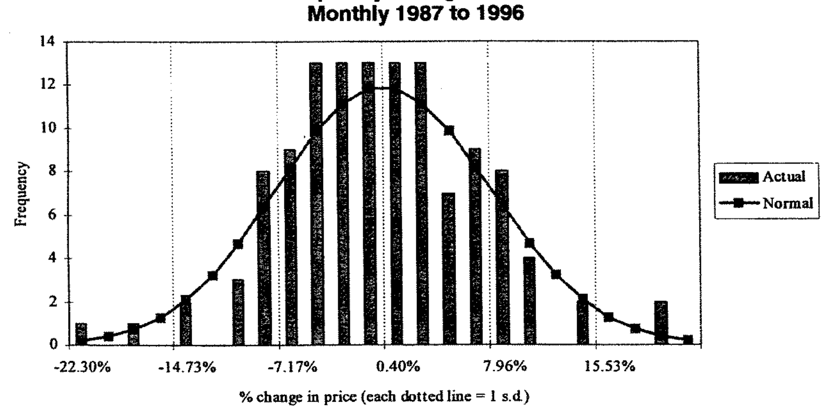
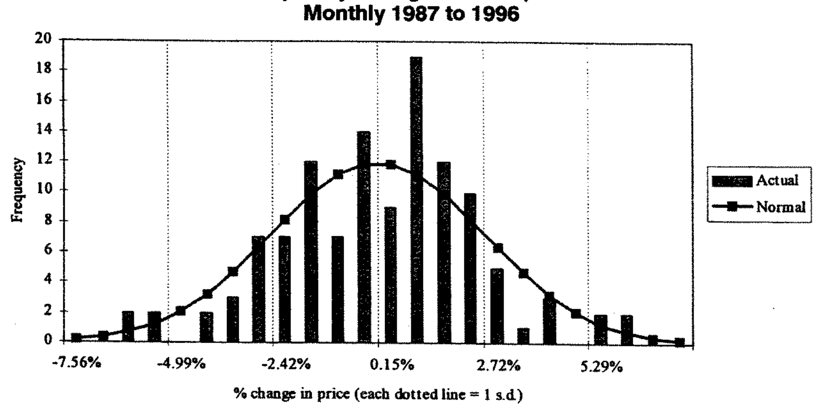
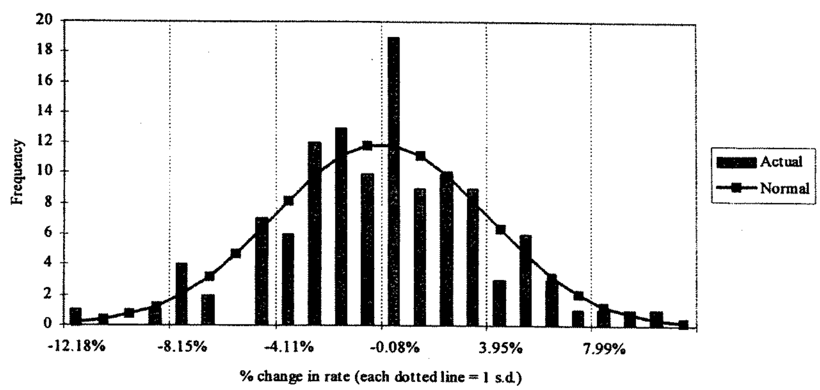


Exhibit 16.48—*continued*

Frequency Histogram—AUD/USD
Monthly 1987 to 1996



Frequency Histogram—US$ LIBOR
Monthly 1987 to 1996



4.2 Outright value at risk

A complicating factor with each of these exposures is that they are interrelated.[ii] For example, a higher oil price inscreases US$ revenues which increases the A$/US$ foreign exchange exposure. Likewise, a lower interest expense will increase net US$ receipts and increase the A$/US$ foreign exchange exposure. When calculating the VAR the impact of these offsetting exposures needs to be incorporated. For our purposes the undiversified VAR will be calculated on an annual basis and it will assume all markets prices move against GIANT's position. The assumed movement in price from current market levels is set out in *Exhibit 16.49.*

ii. Note this is the exposure not the risk factor (that is, the market prices) which determine the market risk of each exposure.

Exhibit 16.49

Volatility Factors

Price	Current	Volatility	Adjusted
WTI	$ 20.00	43.24%	$ 11.35
A$/US$	0.75	14.69%	0.8602
LIBOR	6.00	23.05%	7.38

These price changes represent first estimates only. For example, the oil price has exhibited strong mean reverting behaviour and, based on the last ten years a sustained price movement below US$12/barrel is unlikely. However, as a first estimate it is reasonable.

Using these volatility figures an estimate of the undiversified VAR is set out in *Exhibit 16.50.*

Exhibit 16.50

Undiversified VAR (85% Confidence)

Year	Forecast Profit	Adjusted Profit	VAR	% earnings	% capital
Dec-97	51,415,415	6,671,679	44,743,736	87%	33%
Dec-98	57,887,077	9,269,827	48,617,250	84%	36%
Dec-99	62,912,663	11,218,178	51,694,485	82%	38%
Dec-00	68,264,945	13,306,242	54,958,703	81%	40%
Dec-01	73,962,896	15,542,355	58,420,541	79%	43%

The correlations from *Exhibit 16.46* indicate that the correlations between all three market prices is quite low. That is, a substantial drop in the oil price will not always be accompanied by a sharp rise in the exchange rate and US$ LIBOR.[iii] This lack of any strong negative or positive correlation can lead to a significant diversification benefit when estimating GIANT's VAR.

While a diversification benefit exists across all three market risks, GIANT wishes to manage its interest rate risk independently of the oil price exposure. Accordingly, it looks for diversification benefits arising from the oil price and A$/US$ because it can enter into oil price derivative contracts which hedge just the US$ oil price or the A$ oil price (that is, hedging both the US$ price and the A$/US$ exchange rate).

iii. Interestingly the economic argument would generally be a lower oil price should lead to a decline in interest rate and, potentially, a decline in a commodity currency such as the A$.

The diversification benefit can be seen in *Exhibit 16.51* which illustrates the volatility of the US$ oil price and the A$/US$ exchange rate separately and then the A$ WTI price. On an undiversified basis the 95% volatility is 13.54% higher than the diversified A$ oil price.

Accordingly in 1997 the VAR is approximately $6 million less than the undiversified VAR figure.

Exhibit 16.51

Annualised	67% Price Volatility	95% Price Volatility
US$ WTI	26.20%	43.24%
A$/US$	8.91%	14.69%
A$ WTI	26.90%	44.39%

Undiversified US$ WTI and A$/US$	57.93%
Diversified US$ WTI and A$/US$	44.39%
Difference =	13.54%

Chapter 17

Portfolio Simulation: Stress Testing Techniques

by Lance Smith

1. INTRODUCTION

The term ''stress testing'' refers to an array of risk management techniques that have been developed particularly for portfolios of derivative securities. These go far beyond the traditional methods of simply calculating the current market exposures, and to some extent examine the underlying assumptions of the mathematical models used to calculate these exposures.

In this chapter we will explore a variety of stress testing techniques. These may be classified into two groups: *static* stress testing by input parameters and *dynamic* stress testing by (Monte Carlo) simulation. There are also some interesting connections between these different methodologies, essentially having to do with statistical averages such as volatilities and correlations.

2. WHY STRESS TEST?

Derivative securities exhibit risks that may be inadequately quantified by traditional risk analysis. Greater *leverage* implies a heightened sensitivity to particular market movements, and the potential for such a movement to force liquidation of positions at depressed prices. Such leverage is present not only in options and swaps, but also in long positions that have been financed by repo transactions or purchased on margin.

The introduction of *optionality* or ''gamma'' means that there is the potential for a sudden *acceleration* in exposure to a variety of market factors; for example, the exposure itself may *increase* as the market levels *drop*, thus compounding the potential losses. As a result, it may now become a requirement to rehedge the portfolio due to market movements. A typical instance of this is commonly referred to as ''delta-hedging''. For an equity options trader, delta-hedging is an attempt to maintain a net exposure near zero in his options portfolio by reacting to market movements and trading in the underlying security so as to negate the changes in exposure. If he is net short options, this procedure of delta-hedging will require him to ''buy high and sell low'' as the market moves about. His expectation is that the time decay in his portfolio, which works in his favor, will compensate him for his trading losses incurred in delta-hedging. This implies that the ultimate performance of his portfolio will depend upon the future behaviour of the underlying stock price, and in particular, *patterns of volatility*.

It therefore follows that the risk management of a portfolio containing options (either explicitly or embedded in other securities) should also take into account associated *dynamic trading* issues. The above discussion of delta-hedging, which forms the underpinnings of the Black-Scholes pricing and hedging model, ignores several limitations of the marketplace that do not occur in the theoretical world of stochastic processes.

Liquidity constraints may overwhelm the ability to rehedge this exposure. For example, if an options trader has sold a great many out-of-the-money put options on say, the S&P500, and the market has a gap opening to the downside, he may find himself with a large net long position. Delta-hedging requires him to now sell S&P500 futures against his position. The quantity to be sold may be so large that the actual selling causes the market to drop further, which in turn requires him to sell more, et cetera. This feedback loop can have disastrous consequences. In October of 1987, there were a great many portfolio managers practising ''portfolio insurance''. In effect each was hedging a short out-of-the-money put option. The market drop of Friday, October 16 caused them to go out net long. On Monday morning, October 19, each was ready to sell. The resulting selling pressure touched off such a feedback loop.

Another example is that of *cashflow constraints*. The theoretical models assume that the trader can borrow as much money as he likes whenever he likes, and at an interest rate that has already been fixed. Most of us aren't that lucky. When hedging a derivatives position, there may arise unforeseen borrowing requirements that exceed our credit lines. As a result we may be forced to liquidate our position at depressed prices (since our trading partners may take advantage of our situation). This has happened to more than one hedge fund recently.

A final example is somewhat philosophical in nature. Option models require a volatility input in order to calculate our exposure. Ideally, this input should correspond to the volatility that will be experienced during the delta-hedging process. Unfortunately, nobody knows in advance what this is going to be. We then have to make a leap of faith just to calculate our risks. Ultimately what this means is that we don't even know with precision what our current exposure really is!

3. TRADITIONAL RISK ANALYSIS

Traditional risk analysis for portfolios focuses on *mathematical* derivatives such as: delta, gamma, duration, convexity, et cetera. These are attempts to quantify exposures and rates of change of exposures at *current levels*. Correlations may then also be used to net exposures to different risk parameters. As discussed above, there is also a dynamic hedging component, usually summarised by a single input number, the *implied volatility*. This single number attempts to encapsulate the cost of delta-hedging the option. The relationship is summarised in the following principle:

The delta-hedging principle

The average P&L (in present value terms) in delta-hedging a portfolio of options is obtained by evaluating the options in the portfolio using the (yet to be experienced) volatility. The P&L dispersion is minimised if the "hedging volatility" is the same as the experienced volatility.

The sensitivity to changes in volatility levels is an additional mathematical derivative, usually referred to as *vega* or *kappa*.

3.2 The role of implied volatility

An option's theoretical value, as well as its risk characteristics (delta, gamma, et cetera) will depend upon the level of implied volatility used in the pricing model. As a result, it is very important to understand the criteria by which an implied volatility is determined. This issue becomes more prominent for a portfolio of options, for trading, risk management and stress testing.

The choice of implied volatilities used by the trader may depend upon the actual strategy being pursued. For example, if the trader is making markets in the options in the portfolio, he will most likely use a volatility surface or "smile". This will ensure that he is hedging the *mark-to-market*. On the other hand, if the portfolio is, say, a portfolio of short dated listed options that is being traded proprietarily (that is, taking a particular view of upcoming patterns of volatility), he may use his expected volatilities to calculate his fair values as well as his exposures.

The risk manager may wish to use an "official" set of implied volatilities to evaluate the risk across a broad collection of portfolios being traded for different purposes. The primary concern here is to ensure that the overall book does not take on risks that may be difficult to hedge under extenuating circumstances, or exposures that may lead to large negative valuations that the firm cannot sustain.

4. STRESS TESTING THE EXPECTED VALUE vs STRESS TESTING THE MARK-TO-MARKET

This latter point leads to two complementary methods of evaluating the risk in a portfolio/book. Should I stress test the potential *mark-to-market* of the portfolio or should I stress test its *expected value*, that is, the value that will be realised if delta-hedged to expiration? The first question may be somewhat difficult to analyse in that it requires an assumption on what the mark-to-market may be after a market event (that is, what will the new volatility surfaces/skews be?). This will also depend upon the liquidity of the securities in the portfolio.

The second question is a little more tractable in that it can be related to potential volatilities, invoking the delta-hedging principle. This approach is certainly a good first step in analysing market risk, but does not capture the "tails", that is the low probability but potentially fatal events. For this reason, the *average* P&L may not be a sufficient measure of potential loss. The firm may also be concerned about a probability of loss greater than some fixed amount.

5. STATIC STRESS TESTING

Static stress tests are performed by pushing the relevant input parameters to unlikely , but possible values and then revaluing the portfolio. For a portfolio of equity options, this might mean imposing gap moves upon each underlying security as well as increasing volatility levels. In essence we are evaluating the expected P&L of the portfolio if the underlying stocks suffer a gap move and we delta-hedge to expiration, experiencing higher volatilities than we expected. Some of the questions that arise are: How do the stocks collectively "gap"? Should we use correlations or betas, or should we assume that under strenuous conditions all stocks move with a beta of one. Alternatively, we could assume that again, under stressful conditions the stocks will move with a correlation of 1.

A more sophisticated version is to evaluate the portfolio under stressful conditions straight out of history. One such way is to introduce a *term structure of volatility*. By this we simply mean that we can use a 30 day volatility for a 30 day option, and a 60 day volatility for a 60 day option. In general, when the market suffers a disturbance, the short term experienced volatility may be very high, but it invariably returns to a lower long-term base level. We can go back into history and record the volatilities actually experienced by the stocks, currencies, interest rates, et cetera, and use these to evaluate a book of derivative securities. In effect, we are calculating the expected P&L of the portfolio, should a similar such historical event occur.

6. MONTE CARLO SIMULATION

In Monte Carlo simulation we generate price "paths" of the underlying risks and examine the portfolio along these paths. For example, in a portfolio of equity options we might generate a path of stock prices; that is, we create a synthetic history of stock prices and see how the portfolio performs, given this particular history. We distinguish between two types of Monte Carlo: *passive* and *active*. In passive Monte Carlo we simply watch the portfolio and bite our nails. In active Monte Carlo we delta-hedge along the way. This latter method is a more reasonable approximation to reality and actually ties in with the underlying mathematical pricing models via the delta-hedging principle. Passive Monte Carlo is frequently used for the *pricing* of some derivative securities, but is less applicable for studying market risk in a portfolio of derivative securities.

According to the delta-hedging principle, the average result from a large number of active Monte Carlo simulations should be obtained by evaluating the portfolio using the simulation volatility. The following example illustrates this point.

7. SAMPLE RESULTS ILLUSTRATING THE DELTA-HEDGING PRINCIPLE

As an example of the delta-hedging principle we consider a portfolio consisting of a short position in 3 month call options on IBM, with a strike price of 125. The initial stock price was 118. The options were sold at an implied volatility of 25%. Unfortunately, the volatility to be experienced in the Monte Carlo simulation is 35%. The expected value (obtained by evaluating the portfolio using a 35% volatility) is then about −\$1,000,000, that is, a loss of \$1,000,000. The simulation consisted of 1,000 paths, each of length 100 days, stepping daily. The results of the simulation are:

Average Value	−\$1,086,826
Worst Path	−\$4,011,357
Best Path	+\$1,423,509
97.5% Confidence Level	−\$2,683,823

Note that the average value is as expected. However the worst path had a loss of an additional \$3,000,000. The 97.5% confidence level is \$1,600,000 below the average.

It is illuminating to study the behavior of a particular path in *Exhibit 17.1* and *Exhibit 17.2*.

Exhibit 17.1

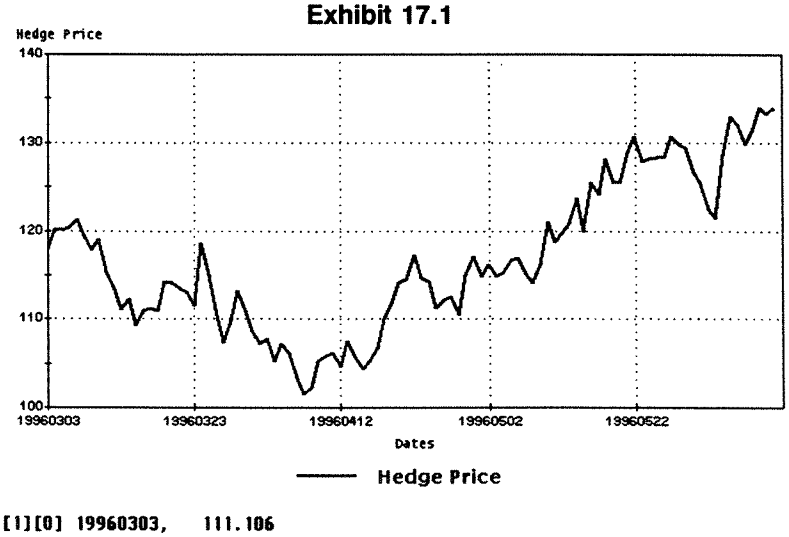
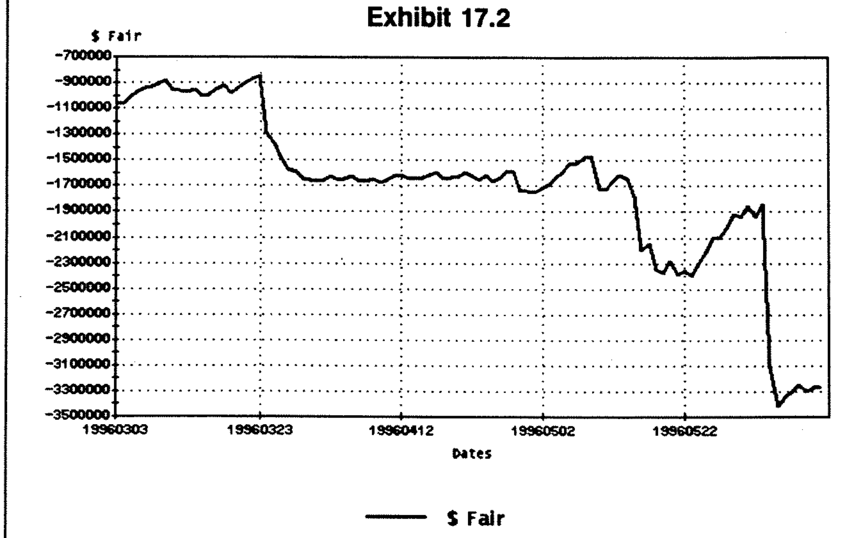
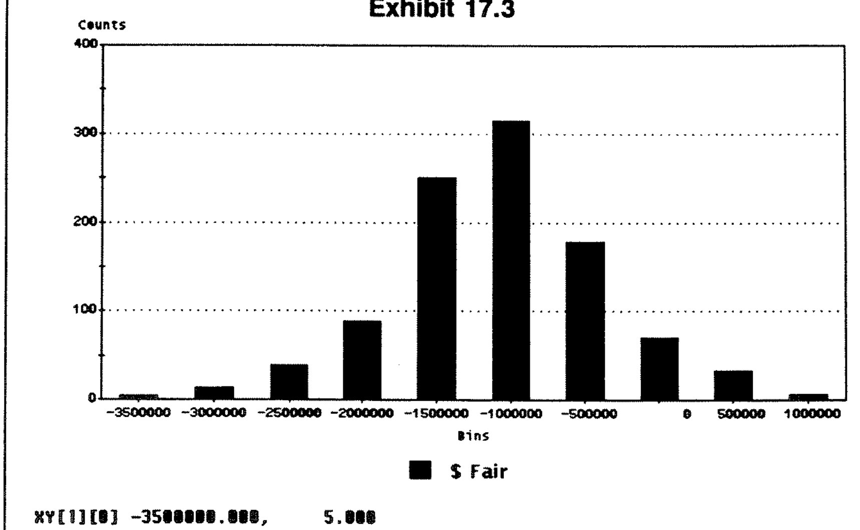


Exhibit 17.2



The above two graphs describe Path 14 in our Monte Carlo simulation. The sudden drops in "$ Fair" are explained by the corresponding "whipsaws" in the stock price. The overall distribution of values for all 1,000 paths are described by the following empirical distribution in *Exhibit 17.3*.

Exhibit 17.3



8. STRESS TESTING AN ACTUAL PORTFOLIO

We will next apply these concepts to a portfolio containing some equity options and their underlying stocks. We first examine the static stress test results.

Hedge SPX implied cash 647.4557

Name	Hedge	Delta cntr	Gamma cntr	US$ theta/day	US$vega 0.1%	US$ fair +rlz
SPX	SPX	9.4	−4.8	21,154	−17,728	1,583,602
INTC	INTC	532.3.	2,016.10	−496	649	−23,310
GM	GM	286.9	3,494.70	−1,411	803	−706
IBM	IBM	−2,670.8	312	−5	3,979	49,982
MO	MO	55.7	1,353.60	−1,177	988	−3,930
EK	EK	−1.5	1,075.10	1,259	1,312	6,315

This is a "book" of six portfolios of options. The first one contains options on the S&P500 index, followed by Intel, General Motors, IBM, Philip Morris, and Eastman Kodak. Reading across, we can see the current exposures represented in "Delta Cntr" (literally, delta expressed in contracts). For the stocks these are share quantities and for the S&P500, futures contracts. Adjacent to this column is the "Gamma Cntr"; that is, the change in Delta Cntr for a one point move in the hedge. The vega column indicates that we are net short S&P500 options, but long individual stock options, each approximately hedged. The current expected value of this book is about $1,600,000, using the term structure of volatility in the following table:

Days	EK	GM	IBM	INTC	MO	SPX
30	27.5	30.6	38.0	40.0	21.6	12.9
91	27.4	28.4	36.1	28.0	19.8	11.6
182	25.0	28.0	28.0	27.3	19.5	11.0
365	24.0	27.5	27.5	27.0	19.0	10.5

We also have the betas of each stock with respect to the S&P500, as well as all pairwise correlations. We will first static stress test by stepping the S&P index by (daily) standard deviations, stepping the stocks accordingly, using their betas. We will also re-evaluate the portfolio at current volatilities and after increasing each volatility by 30% (multiplicatively). For example, 30 day INTC volatility will be increased from 40.0% to 52.0%.

Method 1

	−3 std	Current	+3 std	Maximum Loss
Current vols	$1,270,000	$1,600,000	$1,390,000	(330,000)
130% of current vols	$1,380,000	$1,590,000	$1,370,000	(230,000)

This particular test seems to indicate that an increase in volatility is not a major risk in that the stock volatilities will compensate for the S&P500 volatility. However, if we step the volatility differently, the picture looks quite different. Suppose we assume that as volatility increases, the stock volatility only increases *point for point* with the index volatility.

Method 2

	−3 std	Current	+3 std	Maximum Loss
Current vols	$1,270,000	$1,600,000	$1,390,000	(330,000)
Current vols + 5.0	$ 950,000	$1,130,000	$ 910,000	(690,000)
Current vols + 10.0	$ 600,000	$ 690,000	$ 260,000	(1,340,000)
Current vols + 15.0	$ 245,000	$ 265,000	$ 25,000	(1,575,000)

In this example we do progressively worse for each +5.0 shift increase in volatility.

We can also stress test this portfolio by invoking the volatilities incurred by these stocks during the Iraq-Kuwait crisis that began in August of 1990. Here are the actual volatilities these stocks experienced.

Experienced volatilities beginning August 1 1990

Days	EK	GM	IBM	INTC	MO	SPX
30	37.1	42.0	29.5	52.1	39.9	25.6
91	39.6	30.0	25.5	48.1	30.5	22.0
182	33.5	29.0	23.6	32.0	25.5	19.1
365	30.1	28.0	30.0	30.0	24.8	16.4

The table below shows an expected value of −$210,000. In reality, what occurred was that as the 91 day volatility of the S&P500 increased by 10.4, that of IBM actually *dropped* by 10.6. Certainly the volatilities of the underlying stocks did not keep pace with the S&P500 on a percentage basis (with the possible exception of INTC). This is a feature that is not captured in correlations and other such linear measures of risk.

Method 3

Current	Historical	Maximum Loss
1,600,000	(210,000)	(1,810,000)

9. A MONTE CARLO STRESS TEST

The final stress test we will apply will be a full Monte Carlo simulation in which we increase the individual volatilities by 30%, but maintain current correlations. The correlations used are in the following table:

	EK	GM	IBM	INTC	MO	SPX
EK	1.00					
GM	.10	1.00				
IBM	.06	−.06	1.00			
INTC	.17	.14	.47	1.00		
MO	.33	.34	.22	.12	1.00	
SPX	.52	.32	.43	.52	.51	1.00

The results are quite interesting and square up rather nicely with the historical scenario simulation. The 97.5% confidence interval is found to be $150,000, which is fairly close to the previous simulation. The mean of the run is $1,240,000, which roughly corresponds to the numbers obtained in our first static stress test.

Method 4

Current	Mean	2.5% Level	Maximum Loss
1,600,000	1,240,000	150,000	(1,450,000)

If we interpret the "maximum loss" figure, as the value at risk (VAR), we can summarise the calculations for the four methods employed.

Method	VAR
Method 1 Stepping underlyings by beta to ±3 Std Increasing volatility by 30%	230,000
Method 2 Stepping underlyings by beta to ±3 Std Increasing volatility *linearly* by 15%	1,575,000
Method 3 August 1990 Historical Simulation	1,810,000
Method 4 Active Monte Carlo Increasing volatility by 30% 2.5% percentile	1,450,000

Intuitively, we want a VAR calculation that would incorporate an event such as the August 1990 scenario. Method 1 clearly falls short. Method 2 comes close, but seems somewhat ad hoc. In fact, Method 2 is essentially ''vega bucketing'', that is, analysing the portfolio assuming a linear change in volatility levels across all underlyings. Method 4, the Monte Carlo simulation, also gives similar numbers and has intuitive appeal. The combination of generally increasing volatility by 30%, but allowing for a dispersion of values by considering the lower 2.50% percentile, seems to incorporate the experience of August 1990: volatility was significantly higher, but not for all stocks as shown in *Exhibit 17.4* and *Exhibit 17.5*.

Exhibit 17.4

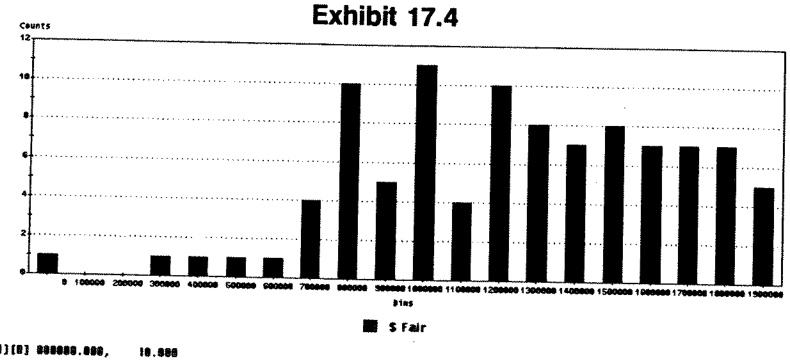
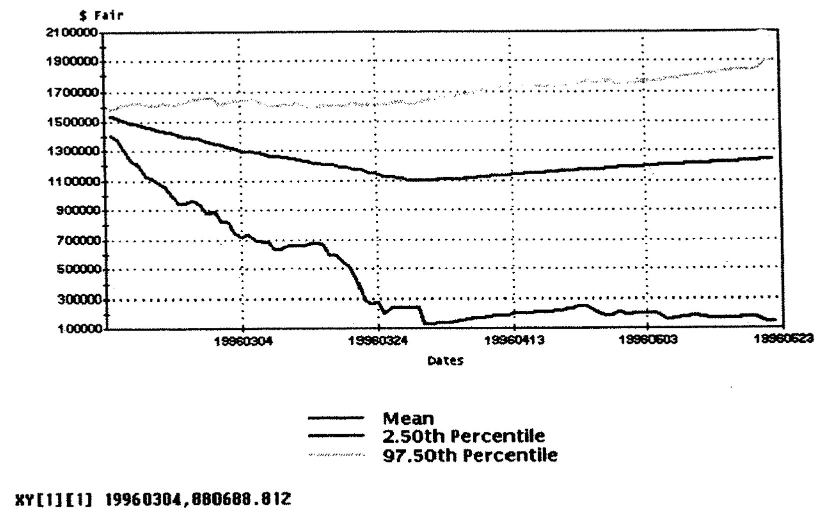


Exhibit 17.5



10. OTHER APPLICATIONS OF MONTE CARLO SIMULATION

10.1 Cash flow considerations

Theoretically derived pricing models usually do not take cash flow requirements into account. For example, suppose I have sold an OTC put on, say, the S&P500 index and hedged it with a short position of 200 Index futures. If the market gaps up 10 points, I will need to come up with $1,000,000 of variation margin, since the OTC position cannot be used against the listed futures margin call. This problem is compounded in a portfolio of OTC options with listed hedges. Another example is if I am running, say, a USD interest rate swap book and hedging with listed (Eurodollar) contracts, a change in the yield curve can produce a large variation margin call. Most pricing models assume that you can borrow as much as you like, whenever you like.

10.2 Liquidity constraints

Theoretical pricing models do not usually incorporate liquidity constraints in the dynamic hedging/pricing algorithm. That is to say, the assumptions of the model assume instantaneous rehedging capabilities in as large quantities as desired, with no market impact. As the stock market crash of 1987 demonstrated, (combined with the then current version of "portfolio insurance"), this is not the case in the real world of trading. In reality you may only be able trade a certain amount when rehedging, without distorting the market or accelerating a trend.

Example

In this example we use active Monte Carlo to investigate the impact of liquidity constraints on an options portfolio. The assumption was that we had sold at the money call options, with one month to expiration on 200,000 shares of stock, expecting a volatility of 30.0%. We then experienced 30.0% in our simulation of 100 paths, so that the average P&L was about $14,042, with a standard deviation of $113,608. We then ran two additional simulations in which we imposed a liquidity constraint of 20,000 shares and 10,000 shares. That is, we were not allowed to trade more than 20,000 shares (10,000 shares) at any one time. The results are summarized below.

Simulation	*No constraint*	*10,000 share limit*	*20,000 share limit*
Average	**14,042**	**6,855**	**4,340**
Standard Dev.	**113,608**	**355,867**	**284,016**
2.50% -tile	**(216,670)**	**(1,114,300)**	**(797,250)**

If we use the lowest 2.50% percentile as a "Value-at-risk" (VAR) calculation, we see that the incorporation of a 20,000 share liquidity constraint yields a VAR that is larger than the unconstrained VAR by a factor of almost 4.0! This additional risk is completely ignored by the other stress tests discussed in this chapter.

11. CONCLUSIONS

Proper stress testing of derivatives portfolios should take dynamic trading issues into account. The delta-hedging principle connects these issues with static stress testing, but this is only an average value calculation that may not adequately measure the impact of unlikely but costly events. Testing by historical scenarios can serve as a good "sanity check" on your risk management methodology. Active Monte Carlo simulation provides the greatest flexibility for analysing dynamic trading issues. Confidence intervals of suitably chosen simulations can serve as good value at risk (VAR) calculations.

Chapter 18

Credit Risk Measurement

by Alan Bustany

1. DEFINITION OF CREDIT RISK

"Credit risk" is the risk that a counterparty to a financial transaction will fail to perform according to the terms and conditions of the contract, thus causing the asset holder to suffer financial loss. The risk exposure to a counterparty can be defined as the amount of positive market value of the portfolio of instruments held with that counterparty at any given time. This exposure can be calculated on a gross basis for each instrument in the portfolio or, where netting across transactions is likely to be legally enforceable, on a net basis. In this section we will assume that the "failure to perform" is a default, and we will generally ignore the possibility of recovery of any portion of the loss from the defaulting counterparty.

A key aspect of the definition of risk exposure is the dependency on time. Credit risk is a function of market risk over the remaining life of transactions with a counterparty. It is this dependency on time that adds complications and computational complexity to the measurement, reporting, and control of credit risk.

2. RISK OF COUNTERPARTY DEFAULT

The economic risk associated with a counterparty is the product of three factors:

(i) the exposure to the counterparty (the main focus of this chapter);

(ii) the probability of default of the counterparty; and

(iii) the potential recovery rate following default.

Typical calculations of economic risk simply multiply a single number representing the credit risk by a single number representing a combination of the probability of default and the likely recovery rate. A more sophisticated analysis would recognise that the probability of default is a function of time (as is the credit risk). The product of the two is therefore also a function of time, which can be calculated by multiplication at each point in time (although, for complete accuracy, the correlation between the default probability and the credit risk over time would have to be taken into account).

For debt instruments the exposure is always close to the outstanding principal, so there is little interaction between the size of the exposure and the event of a default. For derivative contracts, however, the exposure is very variable and there may be significant correlation (positive or negative) with the probability of default. Nevertheless it is rare for such correlation to be taken into account in assessing credit risk.

2.1 Counterparty risk weightings

The simplest approach to calculating economic risk is to assign a "weighting", or probability of default, to each counterparty based on a standard rating agency grade such as those from Moody's Investors Service or Standard and Poor's. Aside from the time-based dependency of the economic risk referred to above, the overall economic risk faced by an organisation is (typically) substantially less than the sum of individual counterparty risks. The usual diversification effects apply, especially since there is normally a very low correlation between default risks of counterparties, although concentrations based on industry, country, and so on can lead to some correlation effects. Nevertheless, the conservative approach of summing the economic risk across counterparties is usually applied.

2.2 Exchange traded versus OTC

Exchange traded derivatives offer credit enhancement compared to over-the-counter transactions directly with a counterparty. The credit risk with the counterparty is effectively replaced by the credit risk of the exchange. The exchange would usually have a better credit rating than a counterparty, and it is further protected by the mechanisms of deposits and margin calls. The effect of this protection is to reduce credit exposure to the exchange to the total value of deposits required, and this does not vary significantly over the term of the transactions. Exchange traded transactions, from a credit risk point of view, are therefore normally treated separately from counterparty transactions.

3. RISK INHERENT IN THE PRODUCT

3.1 Credit risk for derivatives = market risk of replacement

The effect of a counterparty default on a derivative transaction, where there is no exchange of principal, can be completely mitigated by replacing the derivative in the market. At default, therefore, the credit risk of a single transaction is precisely equal to the risk of replacing the transaction in the market. This is typically modelled by the current market value of the transaction, although in low liquidity markets this would usually underestimate the credit risk.

This model also assumes that the capital markets are efficient, that the default does not significantly affect the overall market, and that transaction costs can be ignored. Each of these assumptions also tends to underestimate the credit risk, but these effects are marginal where a reasonable volume of trades in the instrument is effected.

3.2 G30 guidelines: credit risk = current exposure + potential exposure

The G30 report on derivatives[1] states that in assessing credit risk, one needs to ask the following two questions:

1. If a counterparty were to default today, what would it cost to replace the transaction?
2. If a counterparty defaults at some point in the future, what is a reasonable estimate of the potential replacement cost?

These questions are formalised in Recommendation 10 of the G30 report, which states that dealers and end-users should measure credit exposure on derivatives in two ways:

(i) current exposure, which is the replacement cost of derivatives transactions (that is, their market value); and

(ii) potential exposure, which is an estimate of the future replacement cost of derivatives transactions.

It further recommends that potential exposure be calculated using probability analysis based upon broad confidence intervals (for example, two standard deviations) over the remaining terms of the transactions. Note, however, that the report aims to reduce credit risk to a single number for each counterparty, rather than representing the risk as a function of time.

The report further refers to tables of credit risk factors (CRFs) for computation of potential exposure based on a transaction's notional principal. These CRFs are pre-computed, conservative estimates of the maximum potential increase in the value of a transaction. They are typically computed using a Monte Carlo simulation at a 97.7% confidence level and choosing a factor which exceeds the maximum value of the transaction at that level of confidence over time. Such CRFs are computed under specific assumptions about, inter alia, instrument, tenor, market levels, and volatilities. These assumptions help reduce the size and complexity of CRF tables, which are then used in a wide range of contexts (not all of which are strictly valid).

3.3 Settlement risk

Settlement risk arises when there is non-simultaneous exchange of value. It is distinct from the pre-settlement risk on which this chapter focuses. For example, paying an A$ amount in Sydney several hours before receiving a US$ amount in New York gives rise to a settlement risk. Were the counterparty to default during this period, the credit loss would equal the entire US$ amount regardless of the market value of the transaction. Measurement of settlement risk poses no special problems and is not further considered here.

1. *Derivatives: Practices and Principles* (Group of Thirty, Washington DC, July 1993).

4. MEASURING EXPOSURE

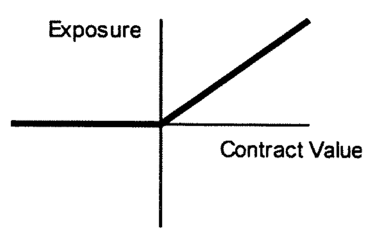
Most market participants use simple rules to estimate a "loan equivalent amount" for derivative transactions. This is generally the current market value of the transaction (an accurate valuation of the current exposure) plus an add-on based on the notional value and credit risk factors for the transaction (a rough estimate of the potential exposure). This section looks at more accurate ways of calculating the potential exposure, and at the statistical assumptions underlying calculations.

Two measures of potential exposure can be used: "expected" exposure and maximum or "worst case" exposure. Expected exposure is the mean of all possible market values over the life of the contract where the market value (and therefore credit exposure) is positive. Worst case exposure is calculated so that, to within some degree of confidence, the actual exposure will not exceed this value. Expected exposure is a useful measure for assessing capital allocation and pricing decisions. Worst case exposure, however, is more useful for assessing credit allocation since it provides a measure of the maximum that could possibly be at risk to a given counterparty. We will therefore focus on worst case exposure for measuring credit risk. Worst case exposure is also more difficult to calculate than expected exposure, although both are complicated by the non-symmetric nature of the exposure.

The exposure on a derivative contract can be regarded as the payoff from an option on the derivative with a strike price of zero. This is because credit exposure only exists when the derivative has positive value as illustrated in the following graph (familiar as the payoff of a call):

Exhibit 18.1

Exposure Against Contract Value



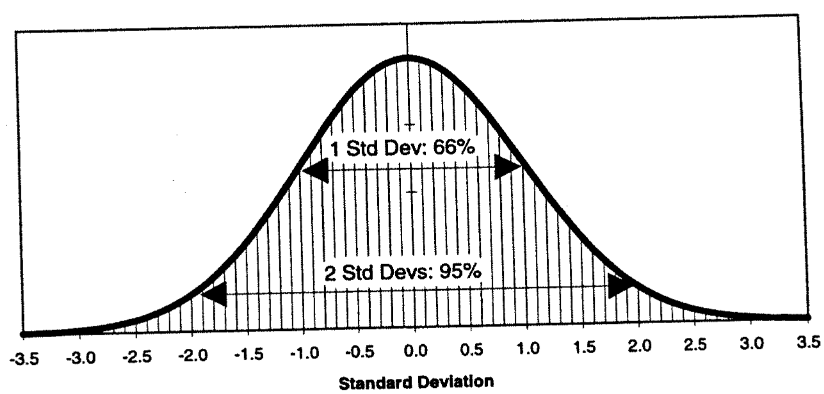
This fact can be useful in analysing credit risk using option valuation models. The zero exposure part of this graph when the contract has a negative value, however, does have an effect on overall credit exposure when netting agreements are in place. Option valuation models are not therefore of much practical use in measuring credit risk on portfolios.

4.1 Statistics applied to credit risk: normal distribution, "fat tails", worst case

A chief assumption made in almost all evaluations of potential exposure is that the underlying market variable is a random variable with a normal distribution about some expected mean. This assumption is made to make calculations tractable and because it is "near enough" to actual behaviour. The normal distribution is used again in converting a statistical value of potential exposure to a confidence level that this exposure will not be exceeded. Thus two standard deviations corresponds to a 95.45% confidence level, three standard deviations corresponds to a 99.73% confidence level, and so on.

Exhibit 18.2

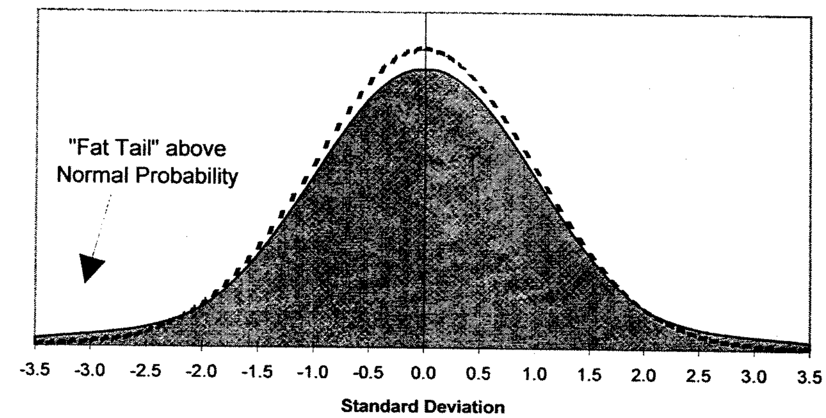
Normal Probability Density



In practice the values taken by almost all market variables can be viewed as near-normal but with "fat tails". That is, for the majority of small deviations from the mean the distribution is almost normal, but the small minority of large deviations occur more frequently than expected. This is of particular concern for credit risk, since worst case exposures occur at the extremes of market variables. Thus we can safely say that 90% of the time the exposure will be less than some value, but it is more difficult to say what value the exposure might have on the other 10% of occasions that, from a credit viewpoint, are more interesting.

Exhibit 18.3

"Fat Tails" Distribution



To address the problems of "fat tails", worst case stress testing (as described in the preceding chapter by Lance Smith) is used to assess credit risk as well as market risk. That is, the potential exposure to a counterparty at points in the future is calculated for all combinations of specific outlying values of market variables. Note that "market variables" typically needs to include the shape of the yield curve at that point in the future for calculation of the exposure to be completed. Industry experience, more than statistics, is used to choose suitable outlying values, so the mathematical significance of the resulting exposures is not clear. But these stress tested values, applied consistently, do help to provide management control of credit risk.

4.2 Simulation—Monte Carlo

A Monte Carlo simulation builds on the fundamental relationship between credit risk and market risk by calculating market values at discrete intervals of time for discrete values of simulated market variables. A probability is associated with each point in the lattice defined by the combinations of time and market variable values. By calculating the exposure for all transactions at each point in the lattice the probability of a certain level of credit risk exposure to each counterparty can be calculated over time. Since the Monte Carlo method depends only on the ability to price a transaction given a set of market variables, and this ability is a pre-requisite of being able to offer a transaction in the first place, such simulations are an obvious way to calculate credit risk. Further details on Monte Carlo techniques can be found in the chapter by Tom Gillespie (Chapter 19).

Although Monte Carlo simulation is a statistically rigorous approach and is applicable across the full range of transactions, it is a practical impossibility to run the requisite number of simulated portfolio market values in a reasonable time, despite enormous increases in computing power in the last decade. A practical method of measuring credit risk exposure in near-real-time is therefore required, hence the tendency to use tables of credit risk factors which can be based on standard runs of Monte Carlo simulations.

4.3 Regulatory measurement

For obvious reasons, regulators have focused on simple, conservative methods for measuring and reporting levels of credit risk. The Bank of International Settlements method focuses on two components: a credit equivalent and a counterparty factor. The credit equivalent can be calculated by either of two methods:

1. the current exposure method (which uses mark-to-market for current exposure and a credit conversion factor to estimate potential exposure); and
2. the original exposure method (which simply uses a credit conversion factor for the whole calculation).

The original exposure method is intended for use when even a mark-to-market calculation is not available, let alone a full calculation of credit exposure over time.

The credit conversion factors for these two credit equivalent calculation methods are:

	% of Nominal Principal Amount	
	Interest Rate Contract	**Exchange Rate Contract**
Current Exposure Method		
Less than one year	nil	1%
One year and over	0.5%	5%
Original Exposure Method		
Less than one year	0.5%	2%
One to two years	1%	5%
Each additional year	1%	3%

The counterparty factor is then applied to the credit equivalent calculated above. Selected counterparty risk weightings are listed below:

	% of Credit Equivalent Amount
Domestic Banks, and Foreign Banks with an original maturity less than 1 year.	20%
Foreign Banks with an original maturity greater than 1 year, and all corporates.	100%

These simple measures can be calculated easily by all market participants. They provide a rough and ready measure for regulators that is consistent across organisations, even if the measure is only an inaccurate approximation of the true credit risks. Such measures are inappropriate for accurate allocation of credit by an organisation.

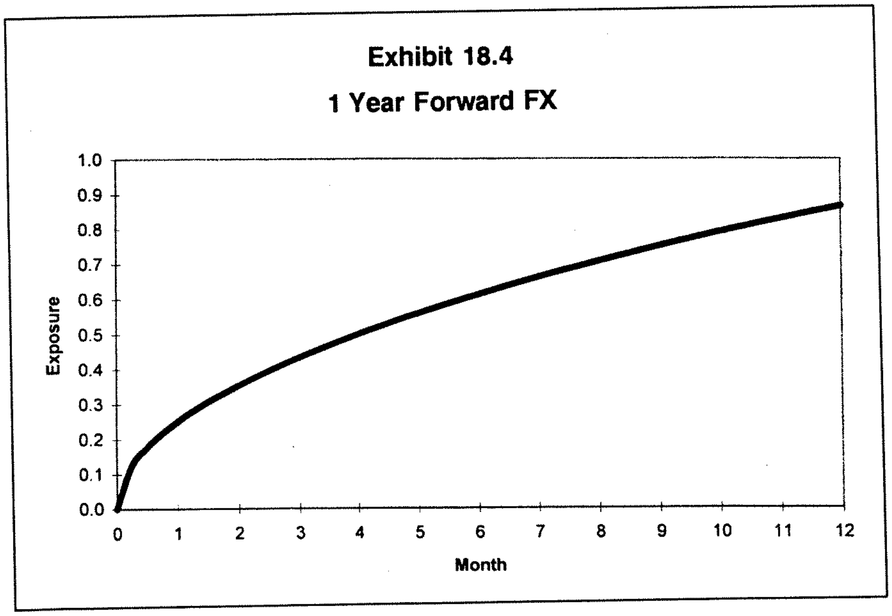
5. TYPICAL EXPOSURE PROFILES OF DERIVATIVE PRODUCTS

For accurate measurement of credit risk, the potential exposure of a transaction or portfolio of transactions with a counterparty should be represented as a profile, measured at some confidence level, at a set of forward times (such as monthly intervals) over the remaining life of the portfolio. This provides a portfolio credit curve which more accurately reflects the credit exposure over time. It also provides the capability to compare the credit curve against a corresponding risk limit curve enabling time-based control over exposure to counterparties.

5.1 Forwards and futures

A forward foreign exchange transaction has a simple monotonic increasing credit curve based on the potential movement of the relevant exchange rate away from the current forward rate for the transaction. As time passes there is an increasing likelihood (at any given confidence level) that the rates will have diverged, but the *rate* of divergence will decrease over time. Assuming a confidence level of say 90% for example, we can model exchange rate movements on a day by day basis to determine at what value we can be 90% sure the exchange rate will not exceed that value on any given date in the future. This exchange rate value translates directly into an equivalent value of our forward contract, which is, in turn, the potential credit exposure on the contract at a 90% confidence level for that date in the future. Performing a Monte Carlo simulation on the underlying exchange rate provides a curve of potential credit exposure.

Ignoring effects from forward rates based on yield curves and other time-based dependencies, the statistical shape of the credit curve is given by exposure increasing as a function of the square root of time. This is simply the effect of assuming that today's exchange rate is a reasonable estimate of future rates, and the "random" movement of the rate will be statistically normal. The typical credit curve for a forward transaction therefore looks similar to the illustration below.



Any forward transaction, including forward rate agreements and over-the-counter futures transactions, will have a similar credit curve analysis based on the movements and volatility of the underlying market variable.

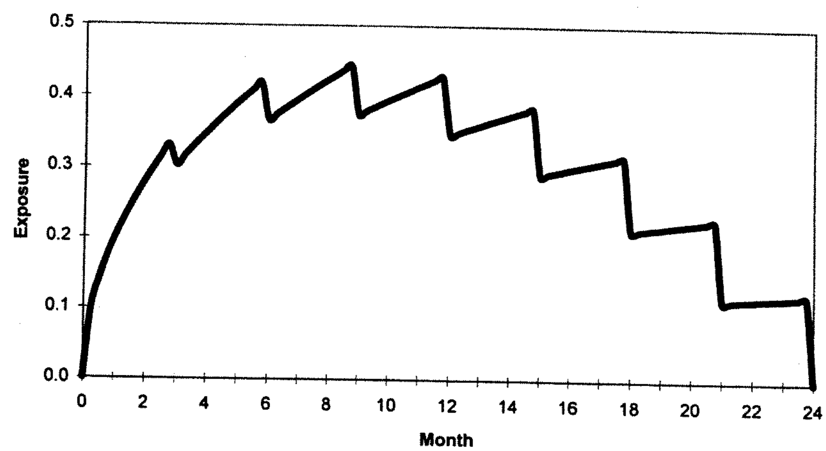
5.2 Swaps

When an interest rate swap is first entered into (at market prices) there is no market value and no credit exposure. If interest rates move in your favour the swap has a positive value, thereby creating an exposure whose expected profile over time is a function of expected interest rate moves and the remaining cash flows on the swap. As for forwards and futures, the expected divergence (at any given confidence level) of the interest rate move will increase over time. The reducing number of cash flows on the swap, however, tend to decrease the exposure in discrete steps as each payment is made. To establish the potential credit exposure curve, we therefore need a simulation of the interest rate to provide the relevant rate at the chosen confidence level, which is then multiplied by the remaining cash flows on the swap (suitably discounted) at that time.

The result is a stepped curve starting at zero, peaking during the life of the swap, and decreasing back to zero as the swap matures. The basic shape of this curve is consistent from swap to swap, but specific values depend on the notional principal, term of the swap, and payment frequencies as well as the usual market conditions. A two year quarterly swap might have a profile as follows:

Exhibit 18.5

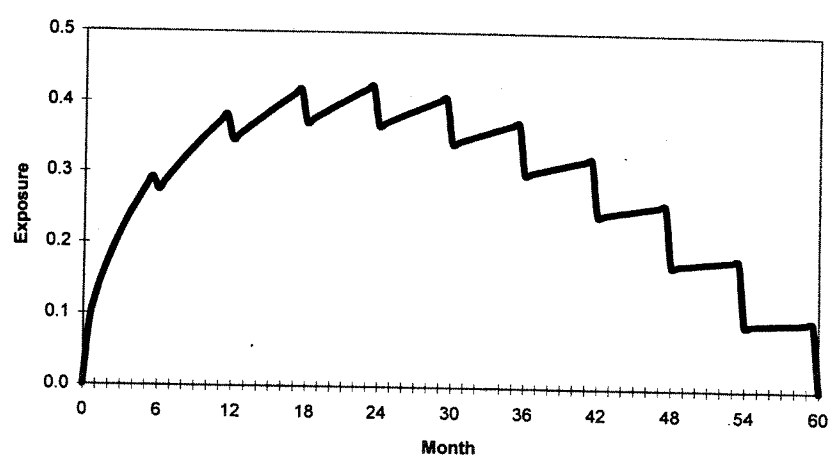
2 Year Quarterly Swap



A five year semi-annual swap might have a profile as follows:

Exhibit 18.6

5 Year Semi-Annual Swap

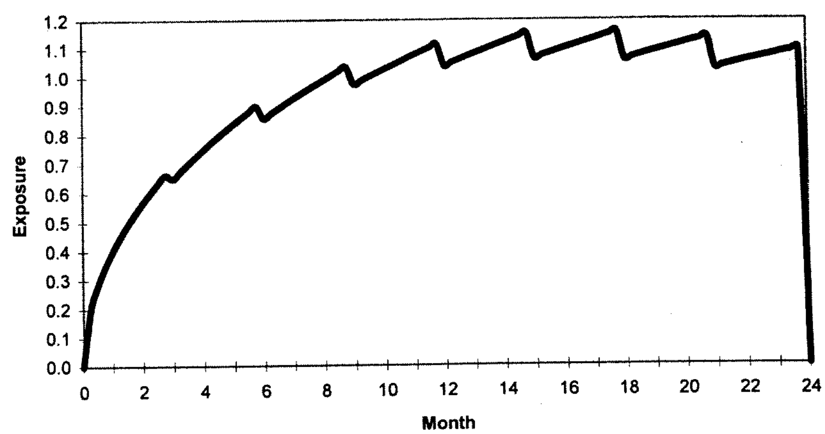


The basic shape of a swap credit profile has a peak at approximately one third of the time to maturity. The precise location and size of the peak is of course dependent on market factors including volatilities and the shape of the yield curve.

Cross-currency swaps which do not include an exchange of principal have essentially the same credit profile as interest rate swaps, but with the added computational complexity of two currencies and two yield curves. Where an exchange of principal is involved, there is effectively a forward FX agreement for the reverse exchange of principal at maturity in addition to the swap's interim cash flows. The credit profile is a combination of the swap and forward, as illustrated below, which is typically dominated by the forward exposure especially as maturity approaches.

Exhibit 18.7

2 Year Cross-Currency Swap (with Principal Exchange)



5.3 Options

The credit risk on an option is one-sided, unlike the situation with forwards and swaps. The buyer of an option typically pays in full when the option is bought. The seller, however, is not required to perform until the option is exercised. This exposes the buyer, but not the seller, to credit risk (which is the inverse of the market risk exposure of the two parties).

Although the value of a simple option is not a linear function of the underlying market variable, it does have extremes at extreme values of the underlying. This is a critical fact when considering simulation models for credit risk which typically follow the highest and lowest possible values (at some confidence level) for, say, the interest rate. Since a simple option will have a corresponding extreme value at these points, this is a valid way of evaluating the credit risk associated with the option. As a result the credit profile of a simple option is again similar to a forward contract.

Combinations of options, and exotic options, however, present a more serious problem for valuation of potential exposure. The pay-off, and therefore the credit exposure for the buyer, can have a peak at *any* value of the underlying market variable including, of course, the current market rate. This is quite unlike other derivatives which, like simple options, typically have extremes of value at extremes of the underlying rate. As a result accurate calculation of the potential exposure of a portfolio of options and exotics with a counterparty requires simulation of all future rates, as well as all time points, to maturity.

The value of an exotic can also be made as sensitive as desired to a particular rate which further complicates the analysis required to establish within the desired degree of confidence what the potential exposure will be. Adding further complications is the fact that the liquidity for exotic options may be limited, resulting in large bid-offer spreads. This affects the underlying assumption throughout this chapter that the credit risk on a transaction can be accurately modelled by a well-defined current market value. Detailed analysis of the valuation and credit handling for exotics is beyond the scope of this chapter.

6. PORTFOLIO EFFECTS

Adding the future potential exposure to the current exposure of a portfolio will almost always overstate the credit risk on the portfolio. Typically, a significant part of the current exposure will settle before the future exposure has time to develop. Secondly, various offsets imply that the exposure on individual transactions cannot logically increase together. Finally, the statistical procedures used for reducing potential exposure on individual transactions to a single number do not permit the simple addition of the resulting numbers—for example adding a peak that occurs at three months to a peak that occurs at two years does not result in the peak of the combined transactions.

The errors involved in the simplifications above are always conservative: that is they are guaranteed to over-estimate the risk. Such conservatism, however, restricts an organisation's ability to do business with a counterparty. It will also lead to poor allocation of risk-adjusted capital. It is therefore a serious problem where credit is in short supply, and it is becoming increasingly important as credit becomes a significant factor in the pricing of transactions.

6.1 Netting

Netting allows the counterparty to a default to add together the positive and negative market values of derivative agreements and either remit the net negative amount or make a claim for the net positive amount. This is a significantly different credit position to the situation in which the gross negative amount is remitted and a claim is made for the gross positive amount, but it depends on the existence and the enforceability of netting agreements. At present netting agreements typically don't apply across the full range of instruments, or across the full range of legal jurisdictions, or to all contracts with counterparties or their subsidiaries. In practice it is therefore necessary to calculate and track nettable exposures as well as non-nettable exposures.

Netting current exposures is a straightforward exercise of summing current market values which can be positive or negative. Netting potential exposures is more difficult since the potential exposure is a statistic summarising the likely worst case at some point in the future. In practice such netting must be embedded in the procedure for calculating potential exposure rather than being applied after the potential exposure is calculated.

6.2 Timing offsets

The fact that potential exposure is a function of time implies that the combination of two exposures with peaks at different times will result in a combined exposure with the following properties:

- the peak of the combined exposure is less than the sum of the peaks of the individual exposures; and
- the combined peak may not occur at the same point in the future as either individual peak.

The fact that a short-term exposure combined with a long-term exposure will produce a combined exposure that is less than the sum of the two individual exposures is referred to as a timing offset. Viewing credit risk as a profile over time, of course, means that the combined exposure is indeed equal to the sum of the individual exposures on a point-by-point basis over the profile.

6.3 Economic offsets

Economic offsets share the properties of netting and timing offsets. The fact that a market variable, say an interest rate, cannot be at both an extreme high and an extreme low at the same time means that a fixed-for-floating swap and a floating-for-fixed swap cannot both have credit exposure at the same time. At one extreme, one swap would have a credit exposure and the other would have no exposure (or could contribute in a netting arrangement). At the other extreme the latter swap would have the exposure and the former none. By combining potential exposures at a point in the calculation where the assumed market variables are known, the benefits of these economic offsets can flow through to the overall exposure.

By viewing potential exposure as a function of a market variable the parallel with timing offsets can be seen. This is particularly so when exotic options are included in the portfolio, since their potential exposure as a function of the underlying market variable can have peaks in the middle of the range similar to the credit profile (over time) of a swap.

6.4 Curve addition

Curve addition is intended to be short-hand for the combination of two credit risk profiles to get the combined profile. Unfortunately, the addition of curves involves more sophisticated mathematics than simple addition if the benefits of the netting, timing, and economic offsets mentioned are to be obtained. Nevertheless, significant improvements over summation of simple numeric credit exposures can be obtained by one or more of:

- netting current exposures (where applicable);
- combining potential exposures point-wise over time; and
- noting which extreme of which market variable drives the potential exposure.

These techniques require representing the credit exposure as a profile in time with various annotations. Note, however, that this approach does not allow for correlations in the instruments or underlying market variables giving rise to the credit exposure. Full evaluation of correlations across multiple variables requires all of these to be incorporated into the initial simulation used to derive relevant values over time.

To be effective as a management tool, the increased sophistication of representing credit exposures needs to be reflected in the representations for reporting and limit setting.

7. MANAGING CREDIT RISK

Credit risk on derivatives cannot be considered in isolation from the credit risks arising elsewhere in an organisation. The credit risk associated with derivatives may be more complex than those associated with, for example, debt instruments, but they need to be aggregated to give a complete picture. This requires the credit risk to be expressed in a comparable manner across all products. Ideally the risk should be translated into the most flexible form (such as that used for representing derivative credit profiles). In practice, a "lowest common denominator" tends to be used, giving inaccurate but conservative "credit equivalents" for derivatives.

Recommendation 11 of the G30 report states that:

> "Credit exposures on derivatives, and all other credit exposures to a counterparty, should be aggregated taking into consideration enforceable netting arrangements. Credit exposures should be calculated regularly and compared to credit limits."

This recommendation emphasises the need to aggregate exposures, but also suggests the two key management criteria of frequent measurement and the imposition of limits for each counterparty.

7.1 Frequency of measurement

The G30 report states that credit exposures should be calculated regularly. In particular, dealers should monitor current exposures daily; they can generally measure potential exposures less frequently. End-users with derivative portfolios should also periodically assess credit exposures. For them, the appropriate frequency will depend upon how material their credit exposures are.

Full measurement of potential exposure on a portfolio basis is a computationally intensive task which cannot, at present, realistically be performed more frequently than daily. A combination of accurate over-night measurement followed by intra-day conservative updates can, however, provide real-time measurements. These real-time credit risk profiles can take into account netting, timing, and economic offsets and allow management against credit limit profiles (as described below). Techniques allowing some form of real-time credit measurement are becoming increasingly important as credit is incorporated into the pricing of transactions.

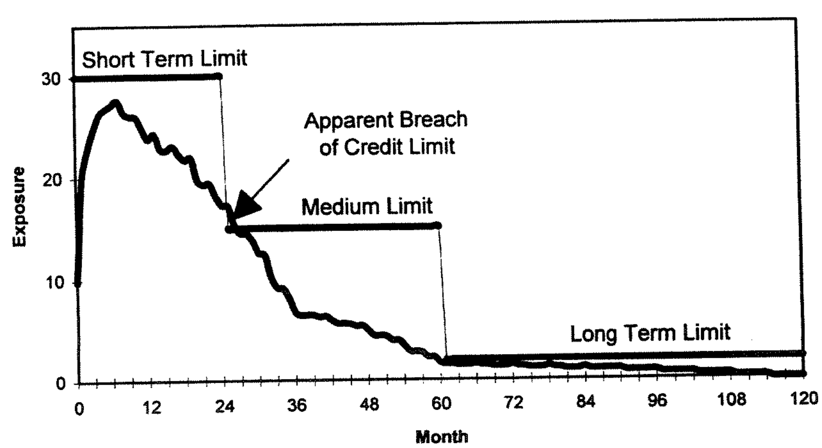
7.2 Limits

Credit limits should be established for all parties with whom an organisation is willing to have a credit exposure. In addition to counterparties this should include brokers, banks, and clearing houses. Typically, organisations impose two or three credit limits for each counterparty: a short-term limit for transactions of less than, say, one or two years; a medium-term limit out to say three or five years; and a long-term limit for greater maturities. They may also impose a limit on the maximum tenor for each counterparty.

Whilst multiple limits can capture some of the time-dependent features of credit risk, a more sophisticated approach is to have a credit limit profile. Such a profile provides a credit limit for the potential credit exposure at each point of time in the future. The chief advantage of this approach is that it avoids the discontinuity that occurs at the boundary between any two specific limits. For example, an organisation's credit profile with a particular counterparty may be perfectly acceptable at all points except for a month or two at the short end of the medium-term period. Given the amount of ''room'' available above the credit risk profile except at particular points, it seems unreasonable to restrict all transactions.

Exhibit 18.8

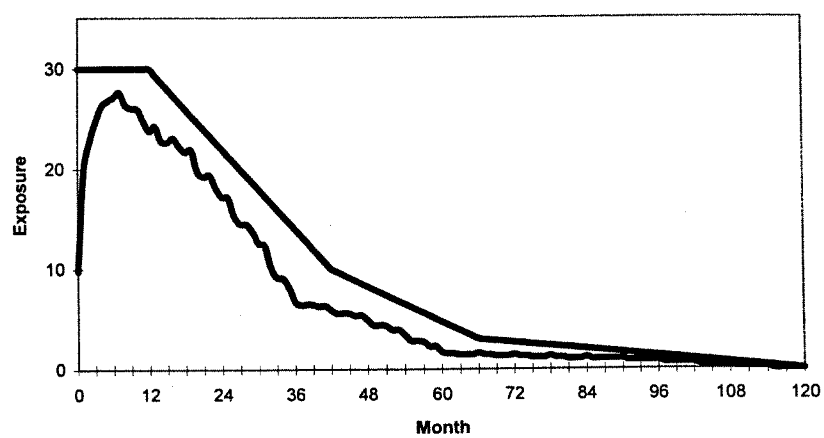
Stepped Credit Limits



The nature of the risk with any given counterparty can be more accurately reflected with a credit limit profile. This can acknowledge certain acceptable peaks with a counterparty and provide limits that gradually decay over time. With such a limit profile, it is less likely that there will be an unrealistic restriction on transactions when there "ought to be" credit available. Equally, a breach of the credit limit will be a "real" breach rather than an artefact of how the risk is measured.

Exhibit 18.9

Credit Limit Profile



The fact that credit risk on derivatives moves with market prices (far more than does the credit risk on non-derivative products) means that a credit limit can be breached without any new transactions occurring. Management procedures to cover monitoring, authorising, and correcting credit limit excesses need to cover the effects of both new transactions and price movements.

7.3 Minimising risk: netting, margin calls, collateral, guarantees

Having analysed and measured credit risk, organisations should take all practical steps to minimise those risks through the use of netting agreements, margin, collateral, or other forms of credit enhancement. We have already referred to the beneficial effects of netting on credit risk exposure (provided a legally enforceable agreement is in place). The use of margins and so on, outside an exchange traded environment, is quite rare but can be particularly useful in dealing with higher risk counterparties.

At present most over-the-counter derivative transactions proceed without either party placing cash margin or any form of collateral as security with the other party. Where such security is provided, it should be noted that the posting of margins or collateral can itself involve a credit risk which should not be left out of the measurement. Margins and collateral can, of course, dramatically change the exposure profile of a transaction. In the case of particularly large, difficult, or unusual transactions, they can be the best means of managing the credit risk.

7.4 Hedging effects

Sophisticated measurement of credit exposure, especially with netting agreements in place, can result in an additional transaction *reducing* the overall credit risk. It can also result in altering the credit risk profile such that it is reduced at one part of the profile whilst being increased at another. These effects can result in a transaction being viewed as a "credit hedge" with one of its purposes being a reduction in credit risk which is of benefit to the organisation.

Given the cost of credit, the effect of a transaction on an organisation's credit risk profile with a counterparty should be taken into account in pricing a transaction. This could enable the organisation to pay slightly more for a credit effective transaction, or to improve the price offered for a product.

Clearly, credit hedges are a secondary consideration to market risk hedges. They are, however, an important consideration especially when credit with a particular counterparty is in short supply. Effective use of such hedges requires relatively sophisticated real-time measurement of credit risk profiles.

Part 6

Mathematical Techniques

Chapter 19

Mathematical Techniques

by Thomas R Gillespie

1. INTRODUCTION

In 1900, one of the most eminent mathematicians of the time, David Hilbert, addressed the second congress in Paris with a talk titled "Mathematical Problems". Hilbert went on to describe 23 of the most important and unsolved problems to mathematics at the time. Most of these problems are still unsolved and hundreds have replaced the few for which a definitive answer has been found. In the meantime hundreds of new avenues of research have sprung up and the sum total of mathematical knowledge has exploded. Mathematical finance is one of these new areas of work.

To summarise all of mathematical knowledge to date in a few dozen pages would be impossible to say the least. To most professional mathematicians today, the bulk of mathematical knowledge is as incomprehensible as it is to the lay person. Mathematicians typically have an intimate knowledge of a few related areas as well as the fundamentals, if they can remember enough of their undergraduate lectures.

The aim of this part of the book is to give a brief introduction to mathematics, so that readers of this book will have a starting point to further enquiry. The bibliography is extremely important, as it gives the direction of the next point of reference. Hopefully, this chapter also makes the book more complete and introduces some of the more common mathematical notation and terms. This chapter has been designed to be referred to, while reading the main part of this volume, by people with little or no training in formal university level mathematics but a desire to better understand the notation and formalism of financial mathematics.

For a general reference to mathematics, *Advanced Engineering Mathematics* by Kreyszig[1] is a good start and covers a lot of territory. For an introduction to statistics in particular, Mendenhall, Scheaffer and Wackerly, *Mathematical Statistics with Applications*[2] or Hogg and Craig, *Introduction to Mathematical Statistics*[3] are useful. These books are not alone, a quick browse through an academic bookshop will yield many alternatives, which are probably just as serviceable. Every mathematician will have their favourites.

The only prerequisite for this chapter is a fair grasp of ordinary calculus as taught in most high schools. The reader is expected to know the difference

1. E Kreyszig, *Advanced Engineering Mathematics* (John Wiley and Sons, 1993).
2. W Mendenhall, R Scheaffer and D Wackerly, *Mathematical Statistics with Applications* (4th ed, Duxbury Press, 1990).
3. R Hogg and A Craig, *Introduction to Mathematical Statistics* (5th ed, Prentice Hall, 1995).

between $\frac{\partial}{\partial x}$ and what the symbol $\frac{d}{dx}$ means and how to manipulate it, and so forth. It is also assumed that the reader is familiar with standard mathematical notation such as $\sum_{i=1}^{n}$ and $\prod_{i=1}^{n}$, standard mathematical functions such as exp(x) or e^x, log(x), sin(x), cos(x) and so on. In all following material, the notation log(x) will denote the natural logarithm function, that is, $\log(e^x) = e^{\log(x)} = x$. For readers not familiar with these concepts see any good high school text or for more detail, see Kreyszig.[4] The notation used in this chapter is the standard form used by most writing in financial mathematics, and summarised in Appendix II.

It is one thing to understand the mathematics, the next job is to code it into a computer to give precise numerical solutions to actual problems. Some people wish to apply a recipe approach to this process. Press, Flannery, Teukolsky and Vetterling, *Numerical Recipes in c*[5] is popular with versions in other languages and now in its second edition. Extreme caution must be exercised with this approach as one volume cannot hope to expound all of numerical mathematics, nor cover recent developments. Another problem is that it gives people a false sense of security; nothing is ever as simple as transcribing a programme out of a book. A little understanding of the basic mathematics and numerical techniques is invaluable.

In section 2 a brief introduction to matrix algebra is covered. Matrix algebra is used to simplify the notation and concepts in areas such as portfolio optimisation and linear regression.

Section 3 gives an introduction to probability theory which is the firm basis for all of statistics. Section 4 discusses random variables and distributions and covers the important distributions in mathematical finance such as the normal distribution. Distribution theory provides a formalism in which to describe random variables such as stock price movements or interest rate movements.

Section 5 covers estimation, that is, it explains how best to derive values for parameters from a model given historical data. The classical example of estimation is deriving estimates for volatility from historical price data.

Section 6 covers the estimation of linear models and describes general optimisation issues. Portfolio optimisation is an ideal example of the use of least squares theory and constrained optimisation, where, usually, tracking error must be minimised given constraints on asset weights and total expected returns.

Section 7 describes random processes. Random processes describe the path a random variable such as a stock price will take from the current level going forward in time. Also covered in section 7 is an introduction to stochastic calculus and the derivation of the Black-Scholes equation.

Section 8 covers simulations and the Monte-Carlo method. Monte-Carlo methods are sometimes the only method for valuation of exotic options and

4. Kreyszig, op cit n 1.
5. W Press, B Flannery, S Teukolsky and W Vetterling, *Numerical Recipes in c* (2nd ed, Cambridge University Press, 1992).

always provide a backup method to check valuations from more analytical methods. Section 8 covers the generation of random numbers and some recent advances.

Appendix I is a brief glossary of names and standard terms used throughout the chapter. Appendix II lists all of the notation used in this chapter and the rest of the book. Finally, please refer to the bibliography at the rear of this book as the first point of reference if the content of this chapter is not detailed or specific enough.

Where possible, examples have been given using Microsoft Excel[6] so that readers may replicate results and understand better the mechanics of the problem. While there are better mathematical programming languages, Excel is serviceable and accessible. These sections are enclosed in shaded boxes and separated from the rest of the discussion.

2. MATRIX ALGEBRA

A *matrix* is simply a rectangular array of numbers. A *vector* is a matrix with either one row or one column only. Matrix algebra is also known as *linear algebra.*

In financial mathematics and risk management the main interest in matrices is that they simplify a lot of the tedious and repetitive calculations in a more elegant formalism. Portfolio optimisation theory lends heavily on this formalism as often portfolios are constructed with hundreds of different but correlated securities. See section 6.5 for a simple finance example. In statistics, linear regression and multivariate statistics are ideal examples of this.

Most undergraduate maths texts[7] contain a more complete discussion of matrix algebra, but the information contained below should suffice for the purpose of this book.

2.1 Notation

A matrix is characterised by its size or the number of rows and number of columns in the array. For example consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 7 \end{pmatrix}$$

This is a matrix with two rows and three columns called **A**. The entries of **A** are usually denoted a_{ij} or $[\mathbf{A}]_{ij}$, the element in the ith row and jth column. For this example $a_{2,3} = 7$. The size of **A** is denoted 2 by 3 or 2 × 3, 2 rows and 3 columns. Notice that the name is usually written in bold type face but sometimes it is inferred by the context of the matrix equation.

6. Microsoft Excel is a registered trademark of Microsoft Corporation.
7. For example see Kreyszig, op cit n 1, Ch 7.

In Excel, matrices (also called arrays) can be entered into joining cells of a worksheet. Look up the Excel User's Guide for a description of how to enter and manipulate arrays. It is especially important to remember to use the [Ctrl][Shift][Enter] keys when inputting array formula instead of just [Enter].

2.2 Addition

Matrices can be added only if they have the same number of rows and columns. The addition of two matrices is simply the addition of each corresponding element. For example if **A** and **B** are defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 7 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 7 \end{pmatrix}$$

then

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1+2 & 2+3 & 2+4 \\ 4+1 & 1+3 & 7+7 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 6 \\ 5 & 4 & 14 \end{pmatrix}$$

Also notice that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

In Excel, matrices can be added simply using the + operator. Using the above example enter the **A** matrix into cells A1:C2, the **B** matrix into A4:C:5. Select a range consisting of 3 rows and 2 columns and enter the formula A1:C2 + A4:C:5, press [Ctrl][Shift][Enter] and Excel will calculate the matrix addition. Notice the curly braces "{" and "}" around the formula once it has been entered to indicate an array formula.

2.3 Scalar multiplication

Multiplication by a scalar is element wise, that is $[3\mathbf{A}]_{ij} = 3[\mathbf{A}]_{ij}$. For example

$$\frac{d\log(L)}{d\mu}$$

then

$$\frac{d\log(L)}{d\sigma^2}$$

and so we have the desirable property $3\mathbf{A} = \mathbf{A} + \mathbf{A} + \mathbf{A}$.

In Excel, matrices can be multiplied by a scalar easily using the * operator. Using the above example enter the **A** matrix into cells A1:C2. Select a range consisting of 3 rows and 2 columns and enter the formula 3*A1:C2, press [Ctrl][Shift][Enter] and Excel will calculate the scalar multiplication. Excel uses the * operator to denote element wise multiplication and so in the above example will automatically expand the scalar 3 to a 2 by 3 matrix of 3s.

2.4 Matrix multiplication

The matrix product **C** = **AB** is defined if and only if the number of columns of **A** equals the number of rows of **B**, if **A** is an m × n matrix and **B** is r × p matrix then n = r and the product **C** is of size m × p. The entries of the product are defined to be $[\mathbf{AB}]_{ij} = \sum_{k=1}^{n} [\mathbf{A}]_{ik} [\mathbf{B}]_{kj}$ that is, the *ij*th element of the product is the sum of the element wise multiplication of *i*th row of **A** and the *j*th column of **B**. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 7 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 9 & 4 \\ 3 & 8 \end{pmatrix}$$

then

$$\mathbf{AB} = \begin{pmatrix} 1*1+2*9+2*3 & 1*2+2*4+2*8 \\ 4*1+1*9+7*3 & 4*2+1*4+7*8 \end{pmatrix} = \begin{pmatrix} 25 & 26 \\ 34 & 68 \end{pmatrix}$$

Notice that for this example

$$\mathbf{BA} = \begin{pmatrix} 9 & 14 & 16 \\ 25 & 22 & 46 \\ 35 & 14 & 62 \end{pmatrix} \neq \mathbf{AB}$$

Hence there is usually a distinction made between multiplying **A** by **B** on the right as in the first example and on the left in the second example.

In Excel, the * operator is reserved for scalar multiplication as discussed above. Excel provides a built in function MMULT() for matrix multiplication. See the online documentation. Remember when inputting matrix formulae to press [Ctrl][Shift][Enter] instead of just [Enter].

2.5 Identity matrix

Just as with real numbers there is a special number 1 such that 1a = a there exists an *identity* matrix **I**. The identity matrix **I** is a square matrix that is composed of mostly 0s but with 1s on the diagonal. For example the 3 × 3 identity matrix is

$$\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

and if

$$\mathbf{A}=\begin{pmatrix}1&2&2\\4&1&7\end{pmatrix}$$

then **AI** = **A**. Notice that for **IA** to be defined **I** must be the 2 × 2 identity matrix and again **IA** = **A**.

The identity matrix is an example of a *diagonal* matrix because its only non zero entries are on the diagonal. Often a diagonal matrix is written in shorthand notation as

$$\text{diag}(a_1, a_2 \ldots a_n)=\begin{pmatrix}a_1&0&\cdots&0\\0&a_2&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&a_n\end{pmatrix}$$

Excel provides no built in function to return an identity matrix. For those familiar with Visual Basic, the following code can be used to return an n × n identity matrix.

```
Function MIDENTITY(n As Integer) As Variant
  Dim i As Integer, j As Integer, result() As Double
  ReDim result(1 To n, 1 To n)
  For i = 1 To n
    For j = 1 To n
      If i = j Then
        result(i, j) = 1
      Else
        result(i, j) = 0
      End If
    Next
  Next
  MIDENTITY = result
End Function
```

2.6 Inverse

In real algebra the inverse of a number is well understood and lets us easily solve equations such as 2x = 4. In matrix algebra, the *inverse* of a square matrix **A** is denoted $\mathbf{A}^{-1}$ and satisfies the following equations, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. For example if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},$$

then

$$\mathbf{A}^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

Just as in real algebra the inverse of 0 is not well defined, not all square matrices are invertable for example the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has no inverse. Such a matrix is called singular as opposed to one which has an inverse which is called non-singular. Actually calculating the inverse of a matrix is problematic and tedious, the mechanics of which are not enlightening and beyond the scope of this chapter. Most spreadsheets can invert matrices and well-established computer codes exist [8]

> Excel provides a built in function MINVERSE() to calculate the inverse of the specified matrix. If no inverse exits then the error #NUM! is returned. The function MDETERM() calculates the determinant and can be used to indicate if the matrix is singular. Singular matrices have a zero determinant.

2.7 Solving simultaneous linear equations

Now that all of this formalism has been establish it will let us solve a simple problem. Consider two simultaneous equations

$$3x + 4y = 6$$
$$2x + y = 9$$

A quick substitution in the second equation of $y = \dfrac{6 - 3x}{4}$ gives the solution

to be $x = 6$ and $y = -3$. These equations can be more concisely written as

$$\mathbf{AX} = \mathbf{B}$$

Where

8. See Press, Flannery, Teukolsky and Vetterling, op cit n 5.

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Left multiplying both sides of this equation by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{I}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Calculating the inverse of **A** gives the solution to be

$$\mathbf{X} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -0.2 & 0.8 \\ 0.4 & -0.6 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

As was calculated before. Of course, this example is almost trivial but it can be seen that the dimension of the problem may be simply expanded to any size and basic matrix calculations are the same. To solve a set of 10 simultaneous equations in 10 unknowns would be tedious and error prone, but with matrix algebra almost trivial.

> Excel can of course be used to solve this set of equations. One such calculation would be to enter the **A** matrix in the range A1:B2, the **B** vector in D1:D2 and enter the equation =MMULT(MINVERSE(A1:B2),D1:D2) in the range F1:F2.

2.8 Transpose

The *transpose* of a matrix is sometimes a useful concept. The transpose of a matrix **A** is denoted $\mathbf{A}^T$ or $\mathbf{A}'$, and is simply the swapping of the rows and columns, that is $[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji}$. For example if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 7 \end{pmatrix}$$

then

$$\mathbf{A}^T = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 2 & 7 \end{pmatrix}$$

A bit of algebra can show the important relationship $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

> Excel provides a built in function TRANSPOSE() to calculate the transpose of the specified matrix.

3. PROBABILITY THEORY

Probability theory forms the basis of most of mathematical statistics. A fundamental understanding of the basics and laws of probability is essential to a deeper understanding of statistics.

In financial mathematics, the pricing of most financial derivatives is made by assuming the probability of future events and so inferring a "fair" price to pay today for the derivative. Probability is an important concept to the valuation of most exotic derivatives as the payoffs are usually expressed as a complex series of events or conditions.

3.1 Definitions and axioms

Most people have a good understanding of the concept of *probability*. A fair coin has a probability of $^1/_2$ of landing heads up, a fair dice roll has a probability of $^2/_6$ of showing a 1 or 2 on its upper face, a particular horse has a probability of winning a race, and so forth. A fully rigorous definition of probability is not very illustrative, and so a more heuristic one will be employed to highlight the fundamentals.

The probability of an event A, a subset of a sample space S consisting of finitely many, equally likely points is defined to be

$$\Pr\{A\} = \frac{\text{Number of points in } A}{\text{Number of points in } S}$$

To take the coin tossing example, the sample space of equally likely points is $S = \{1,2,3,4,5,6\}$ and the event of interest is $A = \{1,2\}$ and so trivially $\Pr\{A\} = {}^2/_6$.

The probability of any event satisfies the axioms of probability

1. $0 \le \Pr\{A\} \le 1$
2. $\Pr\{S\} = 1$
3. For two mutually exclusive events A and B (no point of S lies in both A and B)

$$\Pr\{A \text{ or } B\} = \Pr\{A\} + \Pr\{B\}$$

3.2 Conditional probability and independence

The *conditional probability* is the probability that an event B will happen given another event A has already happened. This probability is written $\Pr\{B|A\}$ and pronounced "probability of B given A". This conditional probability is defined to be

$$\Pr\{B|A\} = \frac{\Pr\{B \text{ and } A\}}{\Pr\{A\}}$$

and so we get the identity

$$\Pr\{B \text{ and } A\} = \Pr\{B|A\}\Pr\{A\} = \Pr\{A|B\}\Pr\{B\}$$

Two events A and B are defined to be independent if

$$\Pr\{B \text{ and } A\} = \Pr\{A\}\Pr\{B\}$$

or equivalently

$$\Pr\{B|A\} = \Pr\{B\} \text{ and } \Pr\{A|B\} = \Pr\{A\}$$

Conditional probability is an important concept in financial mathematics. The price for a particular share today is always implied in any statement or assumption of that share price tomorrow, next week or next year.

3.3 Counting events

Some notation and concepts are useful for counting events, so that one may calculate probabilities.

The number of ways of arranging n distinct objects in order (without replacement) is

$$n(n-1)(n-2)...(2)(1) = n!$$

This is easily shown by observing that there are n choices for the first place, n-1 for the second as one has already been placed, n-2 for the third and so forth.

For example, if we are to consider a 5 horse race there are $5! = 120$ ways the 5 horses can be placed.

This simple construction can be used to show that the number of ways of arranging only k of n distinct objects in order (without replacement) is

$$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

To take our example of a 5 horse race, the number of different combinations for the top 3 places is $\frac{5!}{(5-3)!} = 60$.

If we allow replacement then it is easy to see that the number of ways of ordering only k of the n objects is

$$n \cdot n \ldots n = n^k$$

For example, if 10 tosses of a coin are to be considered there are 2^{10} different combinations of the sequence of heads and tails.

If the order of the k outcomes is unimportant, the number of ways of arranging only k of n distinct objects (without replacement) is

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)\ldots1} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Sometimes $\binom{n}{k}$ is pronounced "n choose k". In the five horse race example the number of combinations for the top 3 places (where order is unimportant) is $\binom{5}{3} = 10$. Calling the horses A, B, C, D and E, it would be a simple matter to write out all 10 combinations ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE. Note that the order is unimportant and so just ABC has been listed instead of ABC, ACB, BAC, BCA, CAB and CBA.

> Microsoft Excel provides some counting functions built in
>
> $$\text{COMBIN(n,k)} = \binom{n}{k}$$
>
> $\text{PERMUT(n,k)} = \dfrac{n!}{(n-k)!}$
>
> FACT(n) = n!

4. RANDOM VARIABLES AND DISTRIBUTIONS

Real variables make a convenient form of abstraction to solve equations such as $ax^2 + bx + c = 0$. Random variables are also convenient to express values that may change.

Financial mathematicians use random variables and distribution theory to describe uncertain stock prices, foreign exchange rates or interest rates. From the assumed distribution the financial mathematician may make various statements about the likelihood of events, the "expected value" of various quantities and finally derive valuation formulae. Distribution theory is used to derive the Black-Scholes equation in section 4.13.

4.1 Definitions

A *random variable* is a real valued function that can take any value from a specified sample space. The value a random variable will take will change with every sampling, test or observation. The frequency that a random variable will take a particular value will depend upon the *probability distribution* that defines the random variable. An alternative definition of the distribution of a random variable X is to define the *distribution function* $F()$ or *cumulative distribution function* such that

$$\Pr\{X \le x\} = F(x)$$

The cumulative distribution function, because it is a probability, must obey the axioms of probability; that is $F(-\infty) = 0$, $F(\infty) = 1$ and $F(x) \geq F(y)$ if $x \geq y$.

The actual distribution is not always known but can be approximated by assuming the form of distribution and estimating the unknown parameters. Most practitioners agree that stock prices are random variables, the actual form of the distribution is not known but section 4.6 below will discuss a popular choice.

4.2 Discrete random variables

A *discrete random variable* is a random variable that can only take a countable number of values. For expediency we will assume that a discreet random variable will only take positive integer values, that is 0, 1, 2, and so forth. A discrete random variable X is defined by its probability function $p(i)$ or p_i,

$$p(i) = \Pr\{X = i\}$$

And so the probability distribution is

$$F(i) = \Pr\{X \leq i\} = \sum_{j=0}^{i} p(j)$$

Each of these probabilities $p(i)$ must satisfy the axioms of probability as discussed in section 2.1; that is, $0 \leq p(i) \leq 1$ for all i, and $\sum_{All\ i} p(i) = 1$.

For example, consider tossing a coin. A random variable X takes the value 1 if a head is observed otherwise it takes the value 0. If the coin is assumed to be fair then the probability function of X will be p(0) = 0.5 and p(1) = 0.5. Consider tossing this coin 10 times and let $X_1..X_{10}$ be the outcomes. These variables are identically distributed and independent with probability function $p_X(i) = 1/2$, $i = 0,1$. If a new random variable Y be the total number of heads, that is $Y = X_1 + .. + X_{10}$, what probability function does Y have? The answer is $p_Y(i) = \binom{10}{i}\frac{1}{2^{10}}$, $i = 0..10$, there are $\binom{10}{i}$ ways of getting i heads and $10\text{-}i$ tails in a sequence and each independent toss has a probability of $1/2$ coming up head or tail.

The random variable Y is said to be a *binomial* random variable and each of the X's is said to be of type *Bernoulli*. Another important type of random variable often found to be a good approximation is the Poisson random variable.

4.3 The Poisson distribution

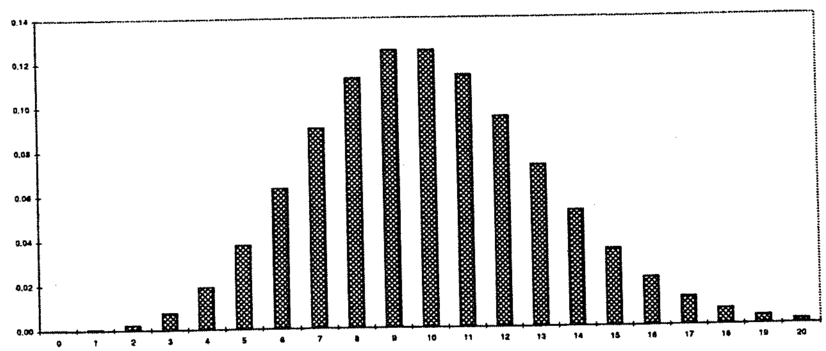
If Y is a Poisson random variable it is defined by a characteristic parameter λ and its probability function is

$$p(y) = \Pr\{Y = y\} = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0,1,2,\ldots$$

The probability function can be represented graphically. For $\lambda = 10$ the probability function takes the following shape in *Exhibit 19.1*.

Exhibit 19.1

Poisson Probability Function



The Poisson distribution is often used as a good approximation to the number of rare events that occur in a fixed unit of time or space. It can be shown that λ is the mean number of events per unit time. One model used to explain stock price moves is the so-called "jump diffusion process", where there are a Poisson number of stock price jumps in a small interval of time.

4.4 Continuous random variables

A *continuous random variable* is a random variable that can take any real value. A continuous random variable X is defined by its probability density function $f(x)$:

$$f(x) = \lim_{dx \to 0} \frac{\Pr\{x \le X \le x + dx\}}{dx} = \frac{dF(x)}{dx}$$

And so the probability distribution function is:

$$F(x) = \Pr\{X \le x\} = \int_{z=-\infty}^{x} f(z)dz$$

The axioms of probability dictate that $f(x) \ge 0$, but as $f(x)$ is not a probability, the function can take values greater than 1.

Continuous distributions are of great use in mathematical finance, as they represent a good framework in which to deal with random stock prices, interest rates and foreign exchange rates.

4.5 The uniform distribution

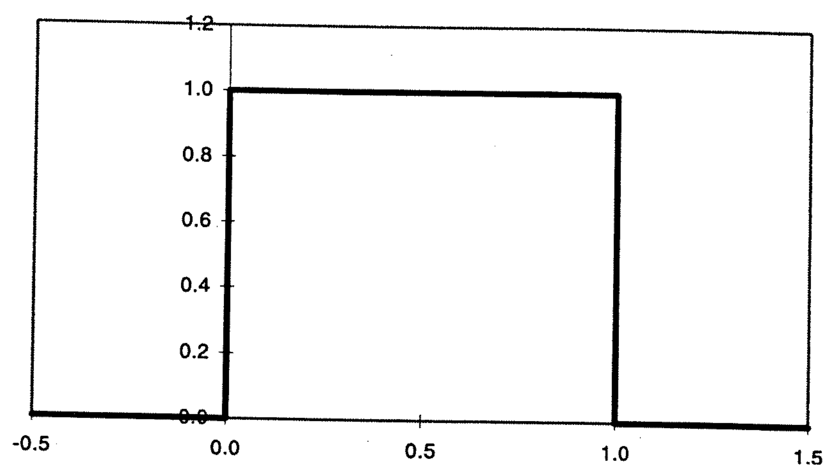
The uniform distribution is probably the simplest of continuous distributions. A uniform random variable U is defined by its probability density function:

$$f_U(u) = \begin{cases} 1 & for\ 0 \le u \le 1 \\ 0 & otherwise \end{cases}$$

The density has been reproduced graphically in *Exhibit 19.2.*

Exhibit 19.2

Uniform Probability Density Function



Uniform random numbers are of great use in the generation of random numbers. Section 8 discusses the generation of random numbers and simulations in some detail.

4.6 The normal distribution

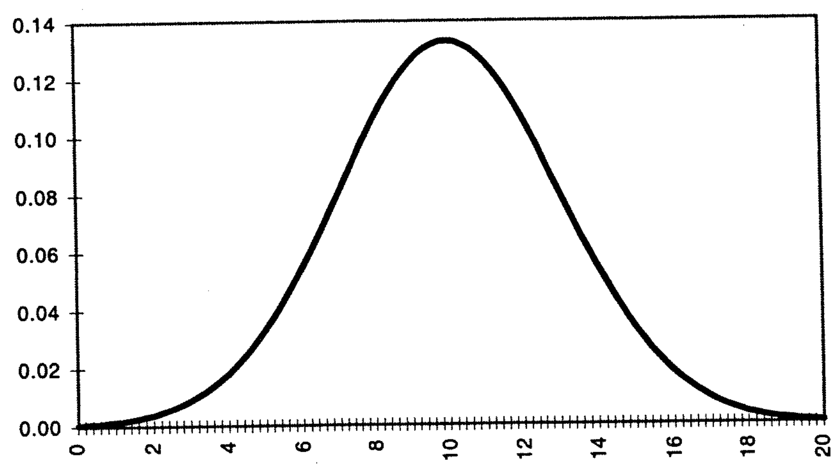
The normal, or Gaussian, distribution is one of the most important in all of statistics. The central limit theorem discussed in section 4.9 shows why this distribution appears again and again in real world applications, approximations and theoretical statistics. The normal distribution is parameterised by a location parameter μ and a spread parameter $\sigma > 0$, the probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

If a random variable X has this probability density function then it is written as $X \sim N(\mu,\sigma^2)$. The sign "$\sim$" is read as "distributed as" and "$N(\mu,\sigma^2)$" indicates the exact form for the distribution. The probability density function for a normal distribution with $\mu = 10$ and $\sigma = 3$ has been reproduced graphically in *Exhibit 19.3*.

Exhibit 19.3

Normal Probability Density Function



A standard normal variable has $\mu = 0$ and $\sigma = 1$ and is the foundation of all normal variables because if $Z \sim N(0,1)$ then $\mu + \sigma Z \sim N(\mu,\sigma^2)$. Another important property of the normal distribution is that it is closed under addition, that is if $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2,\sigma_2^2)$ are independent then $X_1 + X_2 \sim N(\mu_1 + \mu_2,\sigma_1^2 + \sigma_2^2)$.

The cumulative distribution function has no closed form. The cumulative distribution function for a standard normal variable is denoted

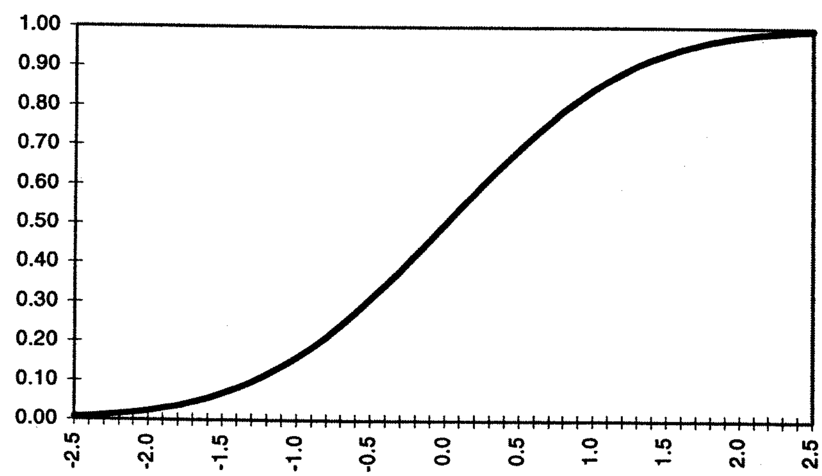
$$\Phi(x) = N(x) = \int_{z=-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

This function is well tabulated in statistics texts and most spreadsheets can calculate these values. See Abramowitz and Stegun[9] for various polynomial approximations.[10] *Exhibit 19.4* below shows a graphical representation.

9. M Abramowitz and I Stegun, *Handbook of Mathematical Functions* (9th ed, Dover, 1972).
10. Equation 26.2.17 of Abramowitz and Stegun, ibid, is the most popular choice having an error less than 7.5×10^{-8}. J Hull, *Options Futures and Other Derivative Securities* (2nd ed, Prentice-Hall International, 1993) advocates an inferior approximation.

Exhibit 19.4

Standard Normal Cumulative Distribution Function



The central assumption of the Black-Scholes[11] equation is that the stock price is log normally distributed, more precisely

$$\log\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r-\tfrac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

Section 4.9 will discuss this assumption further and indicate theoretical and practical reasons for its use.

Excel has built in functions to calculate normal cumulative distribution functions and probability density functions

NORMDIST = The normal cumulative distribution or density function.

NORMINV = The inverse of the normal cumulative distribution.

NORMSDIST = The standard normal cumulative distribution, $\Phi()$.

NORMSINV = The inverse of the standard normal cumulative distribution.

11. F Black and M Scholes, "The Pricing of Options and Corporate Liabilities" (1973) 81 (May-June) *Journal of Political Economy* 637.

4.7 Expected value and variance

The distribution function uniquely determines the distribution of a random variable. To compare two random variables of different types of random variables we can talk about their location and dispersion. These statistics provide descriptive measures of a particular distributions properties. The location or *expected value* of a random variable X is defined to be:

$$\mathrm{E}[X] = \bar{X} = \begin{cases} \sum_{i=0}^{\infty} i p(i) & \text{for a discrete random variable} \\ \int_{z=-\infty}^{\infty} z f(z) dz & \text{for a continuous random variable} \end{cases}$$

This statistic corresponds to our heuristic concept of mean or average.

The spread parameter or *variance* is defined to be:

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - \bar{X})^2\right] = \begin{cases} \sum_{i=0}^{\infty} (i - \bar{X})^2 p(i) & \text{for a discrete random variable} \\ \int_{z=-\infty}^{\infty} (z - \bar{X})^2 f(z) dz & \text{for a continuous random variable} \end{cases}$$

Sometimes a more convenient measure of spread is the *standard deviation*, which is defined to be:

$$\mathrm{StDev}[X] = \sqrt{\mathrm{Var}[X]}$$

For continuous distributions, neither the expected value nor variance need exist. This is a technical point and such cases are rare in practice.

Also, the expected value of a function $g(\cdot)$ of X can be defined as:

$$\mathrm{E}[g(X)] = \begin{cases} \sum_{i=0}^{\infty} g(i) p(i) & \text{for a discrete random variable} \\ \int_{z=-\infty}^{\infty} g(z) f(z) dz & \text{for a continuous random variable} \end{cases}$$

For the normal distribution a little bit of integration can show that if $X \sim N(\mu, \sigma^2)$ then $\mathrm{E}[X] = \mu$ and $\mathrm{Var}[X] = \sigma^2$.

For the Poisson distribution, it can be shown that if $X \sim Poisson(\lambda)$ then $\mathrm{E}[X] = \mathrm{Var}[X] = \lambda$. Hence λ represents the mean number of jumps or events per unit time.

4.8 Higher moments

It is useful to define some more general moments, the nth moment about the origin is defined to be

$$\mu'_n = \mathrm{E}[X^n]$$

and the nth central moment to be

$$\mu_n = \mathrm{E}[(X - X)^n]$$

A few special functions of these central moments are *skewness* and *kurtosis*, defined as

$$\mathrm{Skew}[X] = \frac{\mathrm{E}\left[\left(X - \overline{X}\right)^3\right]}{\left(\mathrm{E}\left[\left(X - \overline{X}\right)^2\right]\right)^{3/2}}$$

$$\mathrm{Kurtosis}[X] = \frac{\mathrm{E}\left[\left(X - \overline{X}\right)^4\right]}{\left(\mathrm{E}\left[\left(X - \overline{X}\right)^2\right]\right)^2} - 3$$

One particularly useful expectation is the *moment generating function* defined as

$$m(t) = \mathrm{E}[e^{tX}]$$

A bit of calculus will show that

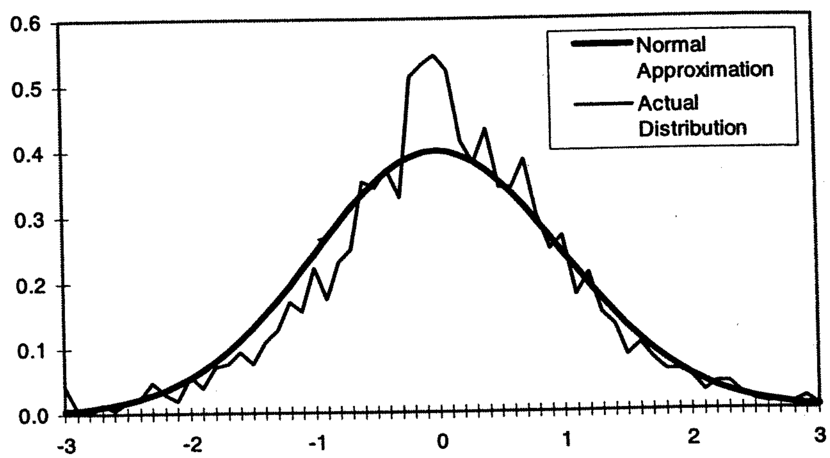
$$\mu'_n = \mathrm{E}\left[X^n\right] = \left.\frac{d^n m(t)}{dt^n}\right|_{t=0}$$

and hence the name. Section 4.10 tabulates most standard probability distributions, their expected value, variance and moment generating function.

For the normal distribution both skewness and kurtosis are 0. Stock prices are commonly described as heavy tailed or *leptokurtic*, which indicates that the observed kurtosis is greater than that of a normal distribution, 0. *Exhibit 19.5* below shows the daily and return distribution for the Australian All Ordinaries Index and the normal approximation.

Exhibit 19.5

All Ordinaries Index Standardised Return Distribution



Notice that the actual daily returns have a narrower distribution with heavier tails. For this sample the estimate for skewness is −0.608 (the distribution is mildly tilted to the left by the outliers) and the estimate for kurtosis is +6.124 (the empirical distribution has significantly narrower body and fatter tails). This observed skewness and kurtosis impacts on option pricing and is thought by most practitioners to explain the *volatility smile*.[12] The observed skew and kurtosis have been used by Jarrow and Rudd[13] to derive correction terms to the Black-Scholes formulae and so take into account observed departures from normality.

Microsoft Excel has built in functions to calculate sample moments. These functions are discussed at the end of section 5.2 on sample estimates.

12. The volatility smile is a description of the implied volatilities observed in most options markets. Out-of-the-money and in-the-money options trade at higher volatilities than at-the-money options and so giving the smile shape when plotted against strike.
13. R Jarrow and A Rudd, "Approximate Option Valuation for Arbitrary Stochastic Processes" (1982) 10 *Journal of Financial Economics* 347.

4.9 Central limit theorem

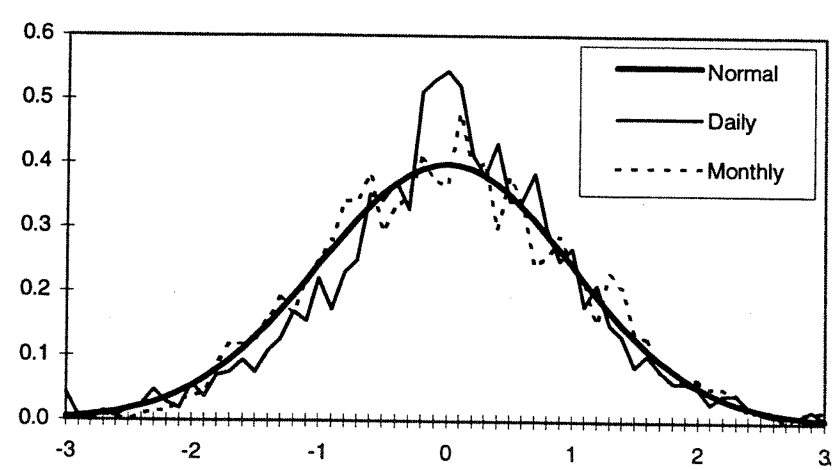
Why is the normal distribution so important in statistics? The reason is if many variables from an unknown distribution are mixed, then the resulting variable tends to a normal distribution.

A formal statement of this theorem[14] would be useful. Let $Y_1, Y_2 .. Y_n$ be independent and identically distributed random variables, with mean μ and finite variance σ_2. The mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ will converge to a normal distribution $N\left(\mu, \frac{\sigma^2}{n}\right)$. The reader is referred to Feller's excellent monographs *An Introduction to Probability Theory and Its Applications*, Volumes I[15] and II[16] for a more complete discussion of the central limit theorem and proof.

The actual distribution for stock prices over very short intervals such as five minutes may not be known. When these distributions are mixed to give the distribution for one day or one week they become very much like a normal distribution. Using daily and monthly data, *Exhibit 19.6* below shows the standardised return distribution for the Australian All Ordinaries Index over 8 years.

Exhibit 19.6

All Ordinaries Index Standardised Return Distribution



14. For a statement and proof of the full central limit theorem see W Feller, *An Introduction to Probability Theory and its Applications* (Vol II, 2nd ed, Wiley, 1971). What follows is a special case of the more general theorem but should suffice for our purposes.
15. W Feller, *An Introduction to Probability Theory and its Applications* (Vol I, 3rd ed, Wiley, 1968).
16. Feller, op cit n 14.

It is the central limit theorem which guarantees[17] the fact that longer and longer averages tend to a normal distribution. The normal distribution is a good approximation to the more complex but unknown stock return distribution. For this reason the Black-Scholes[18] formula is still widely used.

4.10 Common probability distributions

Some common discrete probability distributions are shown in *Exhibit 19.7*.

Exhibit 19.7

Common Discrete Probability Distributions

Name	Probability Function $p(y)$	Mean	Variance	Moment Generating Function
Bernouli	$\begin{cases} p & \text{if } y=1 \\ 1-p & \text{if } y=0 \end{cases}$	p	$p(1-p)$	$pe^t+(1-p)$
Binomial	$\binom{n}{y}p^y(1-p)^{n-y}$ $y=0,1,\ldots,n$	np	$np(1-p)$	$\left[pe^t+(1-p)\right]^n$
Geometric	$p(1-p)^{y-1}$ $y=1,2,\ldots,n$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Poisson	$\frac{\lambda^y e^{-\lambda}}{y!}$ $y=0,1,\ldots$	λ	λ	$e^{\lambda(e^t-1)}$
Negative Binomial	$\binom{y-1}{r-1}p^r(1-p)^{y-r}$ $y=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

Source: Mendenhall, Scheaffer and Wackerly[19] and Abramowitz and Stegun.[20]

Some common continuous probability distributions are found in *Exhibit 19.8*.

17. Given a few regulatory conditions.
18. Black and Scholes, op cit n 11.
19. Op cit n 2.
20. Op cit n 9.

Exhibit 19.8

Common Continuous Probability Distributions

Name	Probability Density $f(y)$	Mean	Variance	Moment Generating Function
Uniform	$\frac{1}{\theta_2 - \theta_1}$ $\theta_1 \le y \le \theta_2$	$\frac{\theta_2 + \theta_1}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$\frac{e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$ $-\infty < y < \infty$	μ	σ^2	$e^{\mu t + \frac{t^2\sigma^2}{2}}$
Gamma	$\frac{y^{\alpha-1}e^{-y/\beta}}{\Gamma(\alpha)\beta^\alpha}$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$ $t < 1/\beta$
Chi-Square	$\frac{y^{\frac{v}{2}-1}e^{-y/2}}{\Gamma\left(\frac{v}{2}\right)2^{\frac{v}{2}}}$ $0 < y < \infty$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	No Closed Form

Source: Mendenhall, Scheaffer and Wackerly[21] and Abramowitz and Stegun.[22]

21. Op cit n 2.
22. Op cit n 9.

Microsoft Excel has many built in functions to calculate the probability density/function, cumulative distribution function and the inverse of the cumulative distribution function:

BETA = cumulative beta distribution function.

BETAINV = inverse cumulative beta distribution function.

BINOMIAL = binomial probability function and cumulative distribution function.

CRITBINOM = inverse cumulative binomial distribution function.

CHIDIST = cumulative chi-squared distribution function.

CHIINV = inverse cumulative chi-squared distribution function.

EXPONDIST = exponential probability density and cumulative distribution function.

FDIST = cumulative F distribution function.

FINV = inverse cumulative F distribution function.

GAMMADIST = gamma probability density and cumulative distribution function.

GAMMAINV = inverse cumulative gamma distribution function.

HYPGEOMDIST = hypergeometric probability function.

LOGNORMDIST = lognormal cumulative distribution function.

LOGINV = inverse cumulative lognormal distribution function.

NEGBINOMDIST = negative binomial probability function.

NORMDIST = The normal cumulative distribution or density function.

NORMINV = The inverse of the normal cumulative distribution.

NORMSDIST = The standard normal cumulative distribution, $\Phi()$.

NORMSINV = The inverse of the standard normal cumulative distribution.

POISSON = Poisson probability function and cumulative distribution function.

TDIST = cumulative Student's t distribution function.

TINV = inverse cumulative Student's t distribution function.

WEIBULL = Weibull probability density and cumulative distribution function.

4.11 Bivariate and multivariate distributions

So far the discussion has centred around univariate distributions, probability distributions in one variable. Often in mathematical finance, distributions of more than one random variable are of interest, or *multivariate* probability distributions. A typical example of this is if the joint distribution of the all ordinaries index and the S&P 500 index is of interest because of a suspected causal relationship between the two equity indices.

The discussion below will focus on continuous random variables; a corresponding derivation for discrete random variables would be an easy extension.

Two random variables X and Y are said to have a *bivariate distribution* if a function $F()$ can be found such that

$$F(x,y) = \Pr\{X \le x, Y \le y\}$$

which satisfies the following criteria

1. $F(-\infty,-\infty) = F(x,-\infty) = F(-\infty,y) = 0 \text{ for all } x, y$
2. $F(\infty,\infty) = 1$
3. $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y} \ge 0 \text{ for all } x, y$

The function $F()$ is usually called the *joint distribution function* and the function $f()$ is usually called the *joint density function.*

The *marginal density* of X is defined to be:

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)dy$$

A similar definition can be made for the marginal distribution of Y. The *conditional density* of X given Y is defined as:

$$f_{X|Y}(x,y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

which of course corresponds directly to the definition of conditional probability given in section 2.2.

Two random variables X and Y are said to be *independent* if

$$F(x,y) = F_X(x)F_Y(y) \text{ or } f(x,y) = f_X(x)f_Y(y)$$

The introduction of more than one random variable introduces the concept of an expected value of more than one random variable. The expected value of a function $g()$ of random variables is defined as

$$\mathrm{E}[g(X,Y)] = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} g(x,y) f(x,y) dx dy$$

A special expectation is the *covariance* of two random variables defined as

$$\mathrm{Cov}[X,Y] = \mathrm{E}[(X-\overline{X})(Y-\overline{Y})] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$$

Notice $\mathrm{Cov}[X,X] = \mathrm{Var}[X]$. The *correlation* between two random variables is defined as

$$\mathrm{Corr}[X,Y] = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}$$

If two random variables are independent then it can easily be shown using the definition of independence that $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$ and so

$$\mathrm{Corr}[X,Y] = \mathrm{Cov}[X,Y] = 0$$

The reverse is not true in general, two random variables may be uncorrelated but may not be independent.

The *conditional expectation* of X given Y is defined as

$$\mathrm{E}\left[X \middle| Y = y\right] = \int_{x=-\infty}^{\infty} x f_{X|Y}(x,y)\,dx$$

A *multivariate distribution* is defined as the distribution of a number of random variables $X_1, X_2, .., X_n$ more conveniently expressed as a column vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

The definitions above for joint distribution function, joint density function, marginal densities and conditional densities have analogous versions for the multivariate case.

The *covariance matrix* is defined as

$$\mathrm{Cov}[\mathbf{X}] = \mathbf{E}\left[(\mathbf{X} - \overline{\mathbf{X}})^T (\mathbf{X} - \overline{\mathbf{X}})\right]$$

$$= \begin{pmatrix} \mathrm{Var}\left[X_1\right] & \mathrm{Cov}\left[X_1, X_2\right] & \cdots & \mathrm{Cov}\left[X_1, X_n\right] \\ \mathrm{Cov}\left[X_2, X_1\right] & \mathrm{Var}\left[X_2\right] & \cdots & \mathrm{Cov}\left[X_2, X_n\right] \\ \vdots & \vdots & & \vdots \\ \mathrm{Cov}\left[X_n, X_1\right] & \mathrm{Cov}\left[X_n, X_2\right] & \cdots & \mathrm{Var}\left[X_n\right] \end{pmatrix}$$

4.12 Bivariate normal distributions

The *bivariate normal distribution* of two random variables X and Y is defined by the probability density function

$$f(x,y) = \frac{\exp\left(\dfrac{-1}{2(1-\rho^2)}\left(\left(\dfrac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_x}\right)\left(\dfrac{y-\mu_y}{\sigma_y}\right) + \left(\dfrac{y-\mu_y}{\sigma_y}\right)^2\right)\right)}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

It can be shown that the five parameters that characterise the bivariate normal distribution are

$\mu_x = \mathrm{E}[X]$

$\mu_y = \mathrm{E}[Y]$

$\sigma_x^2 = \mathrm{Var}[X]$

$\sigma_y^2 = \mathrm{Var}[Y]$

$\rho = \mathrm{Corr}[X,Y]$

It can be shown that by substituting $\rho = 0$ in this equation that two uncorrelated bivariate normally distributed variables are also independent.

A simple extension of this definition is appropriate to define the *multivariate normal distribution*.

4.13 The Black-Scholes equation and expected value

The Black-Scholes equation is easily derived with the use of expected value introduced in sections 4.7 and 4.8. In section 4.6 it was stated that the central assumption of the Black-Scholes equation was that the stock price S_t, of a non dividend paying stock, at a time t in the future is distributed as

$$\log\left(S_t/S_0\right) \sim N\left(\left(r - \tfrac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

or a convenient change of variables gives

$$S_t = S_0 e^{\left(r - \sigma^2/2\right)t + \sigma\sqrt{t}Z}, \quad Z \sim N(0,1)$$

Where S_0 is the current stock price, r is the risk free interest rate and σ is the spread parameter commonly known as *volatility*. This statement was made without proof, but *Exhibit 19.6* of section 4.9 and the central limit theorem itself show that the distributional form is reasonable. The seemingly strange choice of mean parameter is easily explained by computing $E[S_t]$,

$$\begin{aligned}
E[S_t] &= \int_{z=-\infty}^{\infty} S_0 e^{\left(r-\sigma^2/2\right)t+\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= S_0 e^{\left(r-\sigma^2/2\right)t} \int_{z=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z^2-2\sigma\sqrt{t}z+\sigma^2 t\right)+\frac{1}{2}\sigma^2 t} dz \\
&= S_0 e^{rt} \int_{z=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z-\sigma\sqrt{t}\right)^2} dz \\
&= S_0 e^{rt}
\end{aligned}$$

Notice that $S_0 e^{rt}$ is simply the arbitrage free price for the forward.

Consider a European call option,[23] whose strike is K and time to expiry is t, the value of the call option at expiry, as a function of the random stock price, is simply

$$\max(S_t - K, 0)$$

If we define the fair price as the discounted expected value, that is

$$C_{K,t} = e^{-rt}\, E[\max(S_t - K, 0)]$$

This last expression may be easily integrated by making the convenient substitution given above

23. A European call option is the right but not the obligation to purchase the underlying asset at a fixed price (strike price), on a certain date (expiry date).

$$C_{K,t} = e^{-rt} \int_{z=-\infty}^{\infty} \max\left(S_0 e^{\left(r-\sigma^2/2\right)t+\sigma\sqrt{t}z} - K, 0 \right) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= e^{-rt} \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \left(S_0 e^{\left(r-\sigma^2/2\right)t+\sigma\sqrt{t}z} - K \right) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Separating the two integrals and completing the square on the first gives

$$C_{K,t} = e^{-rt} S_0 e^{\left(r-\sigma^2/2\right)t} \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2+\sigma\sqrt{t}z} dz$$

$$- e^{-rt} K \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= S_0 \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z-\sigma\sqrt{t}\right)^2} dz$$

$$- e^{-rt} K \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= S_0 \int_{z=-\frac{\log\left(S_0/K\right)+\left(r+\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$- e^{-rt} K \int_{z=-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Using the properties of the cumulative standard normal distribution function $\Phi(\cdot)$ introduced in section 4.6 simplifies the integrals to

$$C_{K,t} = S_0\left(1-\Phi\left(-\frac{\log\left(S_0/K\right)+\left(r+\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)\right)$$

$$-e^{-rt}K\left(1-\Phi\left(-\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)\right)$$

$$= S_0\Phi\left(\frac{\log\left(S_0/K\right)+\left(r+\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)$$

$$-e^{-rt}K\Phi\left(\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)$$

which is the celebrated Black-Scholes equation.

The value of a European put[24] option may similarly be calculated to be

$$P_{K,t} = S_0\left(\Phi\left(\frac{\log\left(S_0/K\right)+\left(r+\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)-1\right)$$

$$-e^{-rt}K\left(\Phi\left(\frac{\log\left(S_0/K\right)+\left(r-\sigma^2/2\right)t}{\sigma\sqrt{t}}\right)-1\right)$$

This derivation of the Black-Scholes equation is called the *risk neutral derivation* because it relies on the risk neutral property for assets $E[S_t] = S_0e^{rt}$ and for options $C_{K,t} = e^{-rt}\,E[\max(S_t - K,0)]$. The Black-Scholes equation is also derived in section 7.7 using stochastic differential equations which are discussed in section 7.

5. ESTIMATION

Given a theoretical model of a process, and a sample of results, the method of *estimation* will dictate how to guess at the values of unknown parameters of the model. In this way we can establish quantitative measures of the theoretical model that fit the observed sample.

24. A European put option is the right but not the obligation to sell the underlying asset at a fixed price (strike price), on a certain date (expiry date).

For example, given that a Poisson model is thought to explain the move in stock prices and some stock price information, estimation will dictate how to guess at the unknown value of the Poisson model λ. Given the normal model for stock price returns, estimation will indicate how to establish the value of the dispersion parameter σ.

In finance, estimation is used to derive actual values for the parameters of models from historical data. One typical example of this is the derivation of volatility estimators which will briefly be discussed in section 5.4.

Hypothesis testing is the process whereby a rigorous framework is set up to determine within a certain bound of error, if a parameter takes a certain fixed value. Such a test would be to test if the mean of a distribution was zero for instance. Estimation theory can also be used to derive *confidence intervals*. Instead of point measures, confidence intervals are ranges that a parameter can take, based upon the sample data. For example, the mean of a distribution may be estimated to lie in the range [−0.1,0.2] with a 95% confidence. This section will not touch on hypothesis testing nor confidence intervals. Most standard statistics texts such as Mendenhall, Scheaffer and Wackerly[25] or Hogg and Craig[26] will cover these areas in detail.

5.1 Statistics and estimation

A *statistic* or *estimator* is defined as a rule or algorithm used to calculate a quantitative measure of a sample. For example, the sample average is a statistic calculated using the rule

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and the sample of observations $x_1, x_2, \ldots, x_n$.

Consider an estimator $\hat{\theta}_n$ of a population parameter θ using n abbreviations, two desirable characteristics of the estimator are

1. It is *unbiased*, $E[\hat{\theta}_n] = \theta$
2. It is *consistent*, $\lim_{n \to \infty} \text{Var}[\hat{\theta}_n] = 0$

There are two principal means for finding estimators, method of moments and maximum likelihood.

Method of moments estimators are based on the sample moments

$$m'_k = \frac{1}{n}\sum_{i=1}^{n} x_i^{\,k}$$

These sample moments correspond to the moments of the distribution μ'_k which can be expressed as functions of the unknown parameters. Using the assumption that the sample moments are good estimates of the actual moments of the distribution, the equations $m'_k = \mu'_k$ are solved for the unknown parameters.

25. Op cit n 2.
26. Op cit n 3.

A *maximum likelihood estimator* is based on the *likelihood* of a sample $x_1,x_2,...,x_n$ which is the joint density of the variables $X_1,X_2,...,X_n$ evaluated at the realised sample $x_1,x_2,...,x_n$. The likelihood is usually denoted $L(x_1,x_2,...,x_n)$. Maximum likelihood estimates are derived by calculating (usually using calculus) the values for the unknown parameters that maximise the likelihood. More often than not, it is convenient to maximise the log likelihood and using the fact that the log() function is strictly increasing.

A few common examples will clarify these points.

5.2 Sample average and standard deviation

Consider n independent and identically distributed variables $X_1,X_2,...,X_n \sim N(\mu,\sigma^2)$. Let $x_1,x_2,...,x_n$ be a sample of these variables. What estimators are useful to estimate the two parameters μ and σ^2.

Consider $\hat{\theta} = x_1$. This is unbiased, as section 4.7 above showed that $E[X_1] = \mu$, but it is not consistent because $Var[X_1] = \sigma^2$ and so $\lim_{n\to\infty} Var[\hat{\theta}_n] \neq 0$.

Consider the method of moments from above and note that

$$\mu'_1 = \mu$$

$$\mu'_2 = \sigma^2 + \mu^2$$

Equating these equations with the sample moments

$$m'_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$m'_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

gives the method of moments estimators

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s'^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

It can be shown that $\bar{x}$ is unbiased and consistent for μ but s'^2 is biased for σ^2. Defining a new estimator s^2 as

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

It can be shown that the estimator s^2 is unbiased and consistent for σ^2. For most purposes s^2 is used in preference to s'^2.

The same estimators $\bar{x}$ *and* s'^2 can be derived using the method of maximum likelihood discussed above. Noting that it is assumed that $X_1,X_2,...,X_n$ are independent, the likelihood can be written as a product of normal probability density function. The log likelihood can be written

$$\log(L) = -\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2} - \frac{n}{2}\log(\sigma^2) - \frac{n}{2}\log(2\pi)$$

Using standard calculus, it can be shown that setting the derivatives $\frac{d\log(L)}{d\mu}$ and $\frac{d\log(L)}{d\sigma^2}$ to zero, the log likelihood, and so the likelihood is maximised by setting $\mu = \bar{x}$ and $\sigma^2 = s'^2$.

> Microsoft Excel provides many estimation statistics as built in functions
> AVERAGE = Sample average $\bar{x}$
> STDEV = Sample standard deviation s
> STDEVP = Population standard deviation s'
> VAR = Sample variance s^2
> VARP = Population variance s'^2
> DEVSQ = Sum of squares of deviations
> SKEW = Sample skewness estimator
> KURT = Sample kurtosis estimator
> CORREL = Sample correlation estimator
> COVAR = Sample covariance estimator
> FREQUENCY = Sample density function sampled at discrete points.

5.3 Robust estimation

In the previous section on maximum likelihood estimation, it was shown that the estimator $\hat{\mu}$, for the mean of the normal distribution $N(\mu,\sigma^2)$ based upon the sample $x_1, x_2, \ldots, x_n$ was the solution to the equation

$$\sum_{i=1}^{n}(x_i - \hat{\mu}) = 0$$

That is, $\hat{\mu}$ was the sample mean. It is easy to see that if one of the sample was abnormally large or small, an *outlier*, it would have a great impact on the value of the estimator. The estimator is not *robust*.

The mean is one of a more general class of estimator. This more general class of estimators are the solutions to

$$\sum_{i=1}^{n}\Psi(x_i - \hat{\mu}) = 0$$

for some predefined function $\Psi(\cdot)$. For example if

$$\Psi(x) = \begin{cases} 1 & if\, x \geq 0 \\ -1 & if\, x < 0 \end{cases}$$

then the resulting estimator $\hat{\mu}$ will be the median. This is a robust estimator, as an outlier will not change the value of the estimator a great deal.

For example consider the samples from a $N(0,1)$ *Exhibit 19.9* below and note that the second column is the same as the first except the last point is an outlier.

Exhibit 19.9

	Sample 1	Sample 2
DATA	−1.191730	−1.191730
	0.696167	0.696167
	0.099973	0.099973
	0.337278	0.337278
	−0.053640	−0.053640
	0.690118	0.690118
	−0.802490	−0.802490
	0.630671	0.630671
	0.421836	0.421836
	1.379594	5.000000
Mean	0.220777	0.582817
Median	0.379557	0.379557

Huber[27] suggests the use of the following $\Psi(\cdot)$ function

$$\Psi(x) = \begin{cases} -a & if\, x < -a \\ x & if\, |x| \le a \\ a & if\, x > a \end{cases}$$

for $a = 1.5$, and solving (usually numerically) the equation

$$\sum_{i=1}^{n} \Psi\left(\frac{x_i - \hat{\mu}}{\hat{\sigma}}\right) = 0$$

where the estimator $\hat{\sigma}$ is a robust one such as

$$\hat{\sigma} = \frac{\text{median } [|x_i - \text{median } [x_i]|]}{0.6745}$$

This is just a brief introduction to a new and developing branch of statistics. For further reference consult Huber[28] or Launer and Wilkinson.[29]

27. P Huber, *Robust Statistics* (Wiley, 1981).
28. Ibid.
29. R Launer and G Wilkinson, *Robustness in Statistics* (Academic Press, 1979).

Microsoft Excel provides many built in functions useful for generating robust statistics
MEDIAN = Sample median
TRIMMEAN = Mean of an interior subset of a sample
MIN = minimum of a sample
MAX = maximum of a sample
PERCENTILE = the *k*th percentile of a sample
QUARTILE = the quartile points of a sample
AVEDEV = average of the absolute deviations from the mean

5.4 Volatility estimators based on historical data

A common estimation problem in financial mathematics is the estimation of volatility from a historical time series, also known as *historical volatility*. This section will discuss a few commonly used estimators for historical volatility as an example of an estimation problem.

Consider 20 days of data in late 1995 for BHP,[30] the prices and returns (as defined by $\log\left(S_i / S_{i-1}\right)$) were

30. Broken Hill Pty Co Ltd, Australia's largest traded stock.

Exhibit 19.10

Price	Return
18.42	
18.18	−0.01311
18.30	0.00658
18.14	−0.00878
18.16	0.00110
18.08	−0.00442
18.22	0.00771
18.20	−0.00110
18.36	0.00875
18.44	0.00435
18.60	0.00864
18.48	−0.00647
18.62	0.00755
18.36	−0.01406
18.24	−0.00656
18.10	−0.00771
18.30	0.01099
18.10	−0.01099
17.96	−0.00776
17.90	−0.00335

Using the model from section 4.6 for the stock price distribution

$$X_i = \log\left(S_i / S_{i-1}\right) \sim N\left(\left(r - \tfrac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right)$$

where τ is the average time between stock price samples.

The standard method for calculating historical volatility is to use the estimator s^2 for σ^2 and assume there are 260[31] trading days per year and so $\tau = 1/260$. Hence the classical estimator for $\hat{\sigma}_c$ volatility is

$$\hat{\sigma}_C = \sqrt{\frac{1}{2} \frac{\sum_{i=1}^{n} x_i^2 - \frac{\sum_{i=1}^{n} x_i}{n}}{n-1}}$$

For this data set $\hat{\sigma}_c = 0.1324$.

This estimator is not quite the maximum likelihood estimator as the mean of the distribution is a small function of the volatility and the constant interest rate r.

Using the techniques of section 5.2 the log likelihood can be written as

31. This figure will vary slightly from country to country, and for different times of the year.

$$\log(L) = -\frac{\sum_{i=1}^{n}\left(x_i - r\tau + \frac{\sigma^2\tau}{2}\right)^2}{2\sigma^2\tau} - \frac{n}{2}\log(\sigma^2\tau) - \frac{n}{2}\log(2\pi)$$

The maximum likelihood estimator for σ is the solution of the equation $\frac{\delta\log(L)}{\delta\mu} = 0$. There are in fact four separate roots of this equation, two are complex and one is negative, and so the only solution of interest is

$$\hat{\sigma}_{MLE} = \sqrt{\frac{2}{\tau}\left(\sqrt{1 + r^2\tau^2 - 2r\tau\frac{\sum_{i=1}^{n} x_i}{n} + \frac{\sum_{i=1}^{n} x_i^2}{n}} - 1\right)}$$

For this data the estimator is $\hat{\sigma}_{MLE} = 0.1322$, not overly different from the classical estimator because the mean term of the distribution $(r - {}^1\!/_2\sigma^2)\tau$ is always much smaller than the standard deviation $\sigma\sqrt{\tau}$. Hence, for the purposes of estimating volatility from a frequent data set, an equally good assumption is that

$$X_i = \log\left({}^{S_i}\!/_{S_{i-1}}\right) \underset{approx}{\sim} N(0, \sigma^2\tau)$$

and so based on this approximate assumption, the maximum likelihood estimator is

$$\hat{\sigma}_{Approx} = \sqrt{\frac{1}{\tau}\frac{\sum_{i=1}^{n} x_i^2}{n}}$$

For this data the estimator is $\hat{\sigma}_{Approx} = 0.1311$.

An alternative is to use a robust estimator as introduced in section 5.3 with this new approximate assumption.

The robust estimator is

$$\hat{\sigma}_{Robust} = \frac{median\left[|x_i|\right]}{0.6745} \times \sqrt{\frac{1}{\tau}}$$

For this data the estimator is $\hat{\sigma}_{Robust} = 0.1842$, this deviation is partly a result of the unrealistically small sample size and partly a result of the non normality of returns.

Often it is assumed that the return distribution is non stationary, the volatility fluctuates as well as the stock price. Management of companies, core business activities and other fundamental reasons indicate that only a short period of historical data should be used to predict future volatility. To accommodate this reality either a short estimation period is used compared to the entire data set. Often it is possible to observe price data for 10 years or more on a daily basis but an estimator based on approximately 2,600 will be very efficient but will be little or no use to predict future volatility.

Another method is to use a weighted average of the returns squared. A popular scheme is the exponentially smoothed volatility

$$\log\left(\frac{P_i}{P_{i-1}}\right)$$

Notice that $w_i = \alpha w_{i+1}$, the most recent data is given the most weight and each preceding piece of data is given slightly less weight. For this data, and choosing $\alpha = 0.97$, the estimator is $\hat{\sigma}_{ES} = 0.1318$. The selection of α will influence the decay of the weights, the closer α is to 1 the more the estimator is influenced by past data and the more stable it is.

More sophisticated estimators are based on open-high-low-close data instead of the close-close data, see Garman and Klass[32] and Parkinson.[33]

6. LEAST SQUARES AND OPTIMISATION

A common problem in mathematical finance is finding the best estimators for a linear model. An example of this sort of problem is fitting a model such as $y = \alpha + \beta x$ where y represents a stock return such as BHP, x is the market return and α and β are constants to be estimated. This is the single index model which is a generalisation of the commonly used CAPM. Section 6.5 discusses another application in mathematical finance.

The broad method used for estimating constants from such models is called *least squares*, the reason will become apparent in the later discussion. Special cases of this technique are called *linear regression and multiple linear regression*. The least squares method will be introduced by discussing these two models in detail.

A discussion of least squares leads naturally to more generic problems of *optimisation* and *constrained optimisation*.

6.1 Linear models

Consider a stochastic model of the form

$$Y_i = \alpha + \beta x_i + \varepsilon_i$$
$$\varepsilon_i \sim N(0,\sigma^2)$$

There are many cases where such a linear model may be appropriate. For example, take the relationship between the returns from a stock such as BHP and the Australian all ordinaries index. For a period of 20 days in late-1995 the stock and index prices were

32. M Garman and M Klass, "On the Estimation of Security Price Volatilities from Historical Data" (1980) 53(1) *Journal of Business*.
33. M Parkinson, "The Extreme Value Method for Estimating the Variance of the Rate of Return" (1980) 53(1) *Journal of Business*.

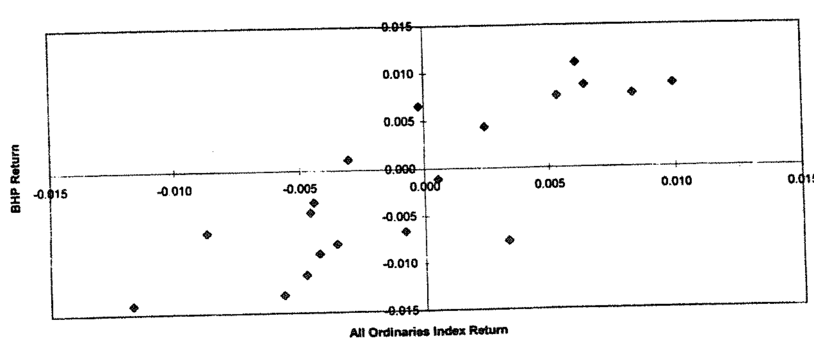
Exhibit 19.11

BHP Price	All Ordinaries Price
18.42	1902.4
18.18	1891.6
18.30	1891.2
18.14	1883.2
18.16	1877.5
18.08	1868.9
18.22	1884.5
18.20	1885.5
18.36	1904.3
18.44	1908.9
18.60	1921.2
18.48	1904.5
18.62	1914.7
18.36	1892.4
18.24	1890.9
18.10	1897.2
18.30	1908.8
18.10	1899.7
17.96	1893.0
17.90	1884.6

The returns (as defined by $\log\left(P_i / P_{i-1}\right)$) plotted against each other with BHP returns on the vertical axis and the all ordinaries index returns on the horizontal axis appear as in *Exhibit 19.12*.

Exhibit 19.12

Return for BHP vs Return for All Ordinaries Index



A linear relationship such as $BHP = \alpha + \beta.AOI$ seems a plausible explanation for this data. There seems to be a number of possible choices because of the random terms ε_i. One commonly used choice for α and β are the least squares estimators.

Consider an estimate for Y_i denoted $\hat{y}_i = \alpha + \beta x_i$, the least squares estimates $\hat{\alpha}$ and $\hat{\beta}$ for α and β are those which minimise the sum of the squares between the actual observed data y_i and the estimate $\hat{y}_i$, that is

$$SS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - (\alpha + \beta x_i))^2$$

The actual minimum point is found by taking partial derivatives with respect to α and β and setting equal to zero

$$\frac{\partial SS}{\partial \beta} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i) \cdot -x_i = -2\left(\sum_{i=1}^{n} y_i x_i - \alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 \right)$$

$$\frac{\partial SS}{\partial \alpha} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i) \cdot -1 = -2\left(\sum_{i=1}^{n} y_i - \alpha \sum_{i=1}^{n} 1 - \beta \sum_{i=1}^{n} x_i \right)$$

Setting both of these partial derivatives to 0 and solving for $\hat{\alpha}$ and $\hat{\beta}$ gives

$$\hat{\beta} = \frac{n\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

These equations are also the maximum likelihood estimators for α and β if the error terms ε_i are assumed to be normal and independent.

Given the estimator for the correlation between the BHP returns and the all ordinaries returns is

$$r = \frac{n\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2} \cdot \sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i \right)^2}}$$

and the estimates for the variance of the BHP returns and the all ordinaries returns are the usual estimates, that is

$$s_x^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i \right)^2}{n}$$

$$s_y^2 = \sum_{i=1}^{n} y_i^2 - \frac{\left(\sum_{i=1}^{n} y_i \right)^2}{n}$$

then the important relationship

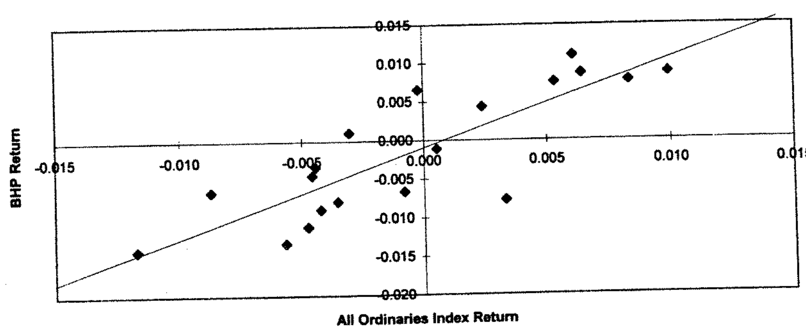
$$\hat{\beta} = r\frac{s_y}{s_x}$$

can be derived. This relationship is often used to calculate the estimator of β from the estimate for the correlation coefficient and vice versa.

Using these equations and the data above, the estimates for α and β are $\hat{\beta} = 1.1503$, $\hat{\alpha} = -0.0009$. Charted with the data as in *Exhibit 19.13* below, it can be seen that this selection of parameters explains the variation in the data reasonably well.

Exhibit 19.13

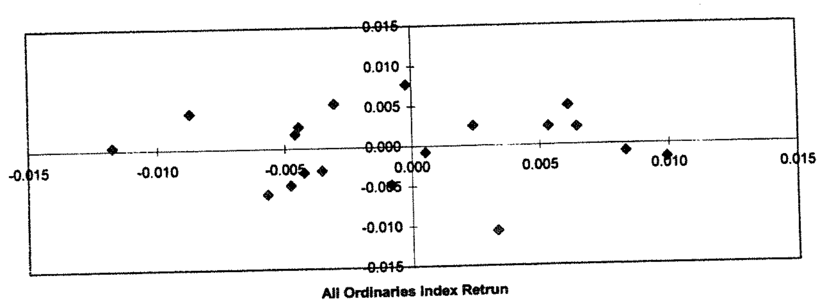
Return for BHP vs Return for All Ordinaries Index



Alternatively, the errors, when plotted against the all ordinaries returns, as in *Exhibit 19.14* below, show no systematic deviation.

Exhibit 19.14

Regression Errors



This is only a brief summary of the major results of linear regression; the full theory derives estimates for the variance of the *Y* estimates, confidence intervals for the parameters and Y estimates, and test statistics for a variety of different hypothesis. The interested reader is referred to Neter and Wasserman,[34] Neter, Kutner, Nachtsheim and Wasserman[35] or any similar book dealing with linear regression.

6.2 Multiple linear models

The theory of multiple linear regression extends the simple theory derived above to more than one X variable. Matrix algebra considerably simplifies the derivation and calculation of the estimators. The methods of section 6.1 above can easily be rewritten using matrix algebra. If we redefine the problem as

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad \beta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon$$

and the sum of squares expression can be rewritten as

$$SS = (\mathbf{Y} - \mathbf{X}\beta)'(\mathbf{Y} - \mathbf{X}\beta)$$
$$= \mathbf{Y}'\mathbf{Y} - 2\beta'\mathbf{X}'\mathbf{Y} + \beta'\mathbf{X}'\mathbf{X}\mathbf{B}$$

Minimising this expression with respect to the vector β yields the estimator

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

This equation may be easily extended to p dimensions, that is

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{p1} \\ x_{21} & x_{22} & \cdots & x_{p2} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{2n} & \cdots & x_{pn} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix} \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon$$

and again the estimators are given by

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

Note that one column of **X** may be 1's as in the simple linear case, and hence give a constant value. Columns of X may be entirely 0's and 1's in which case it is called an indicator variable. Such a use would be in clinical trials where some patients are given a drug and some are not.

Sometimes (**X'X**) may not be invertable; in this case the model has been over specified and one or more of the x variables must be dropped out. A simple case is if two of the columns of **X** are entirely 1's, that is two constant factors. In this case one can easily be dropped.

34. J Neter and W Wasserman, *Applied Linear Statistical Models* (Irwin, 1974).
35. J Neter, M Kutner, C Nachtstein and W Wasserman, *Applied Linear Statistical Models* (4th ed, Irwin, 1995).

Microsoft Excel has a comprehensive multiple linear regression function built in. LINEST will return an array of information such as the estimates themselves, standard deviation estimates for the estimates, the correlation coefficient, and an F statistic. See the comprehensive online documentation for a description of the input and outputs of this function. Excel also provides other associated functions, FORECAST and TREND to calculate values along a linear trend, FDIST and FINV for interpreting the calculated F statistic from LINEST, TDIST and TINV for interpreting the standard deviation estimates for the estimates from LINEST.

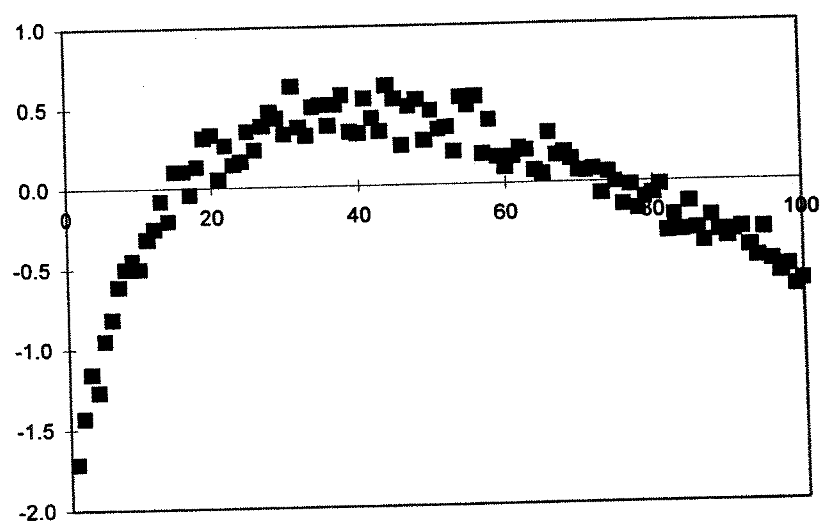
Excel also provides regression functions LOGEST for a regression using log transformed y's and x's, but it is usually more convenient to use LINEST and transform as required.

6.3 Nonlinear models

In more complex situations a linear model may not be appropriate. Consider the error estimates plotted against the original x values as in *Exhibit 19.15 below.*

Exhibit 19.15

Error Terms



Obviously the linear model is not appropriate description of the data and a more complex model may fit better. Sometimes the required model can be rewritten as a linear model. Consider a model like

$$Y_i = \alpha x_{1i}(1 + \beta x_{2i}) + \varepsilon_i$$

$$\varepsilon \sim N(0,\sigma^2)$$

This can be rewritten as

$$\begin{aligned} Y_i &= \alpha x_{1i} + \alpha\beta x_{1i}x_{2i} + \varepsilon_i \\ &= \alpha x_{1i} + \beta' x_{3i} \end{aligned}$$

where $x_{3i} = x_{1i}x_{2i}$ and $\beta' = \alpha\beta$. This is obviously linear in the transformed variables now and standard multiple linear regression may be used to estimate the variables.

More often than not, the equation can not be transformed to a linear one, such as

$$Y_i = \alpha\left(1 - e^{-\beta x_i}\right) + \varepsilon_i$$
$$\varepsilon_i \sim N\left(0, \sigma^2\right)$$

For this kind of problem it is more convenient to return to first principals. The sum of squares to be minimised is

$$SS = \sum_{i=1}^{n}\left(y_i - \alpha\left(1 - e^{-\beta x_i}\right)\right)^2$$

This expression may be minimised either analytically (rarely) or numerically. For a discussion of some numerical methods see section 6.6 below.

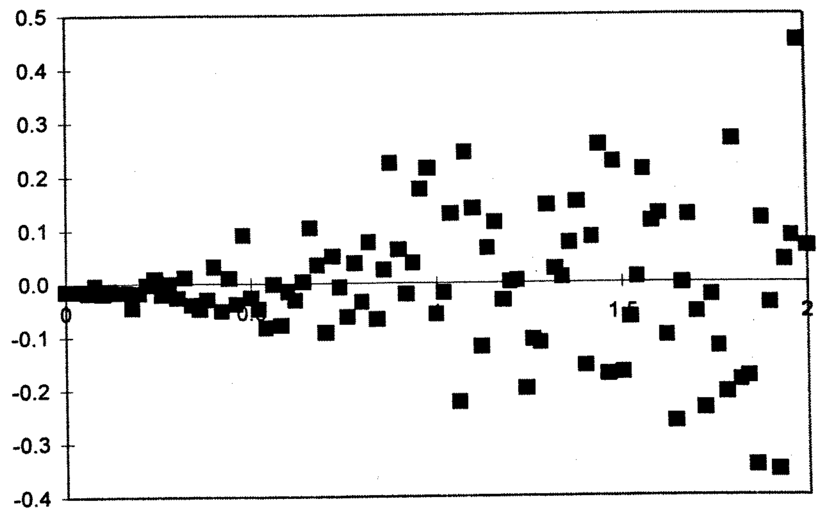
6.4 Robust regression

With ordinary least squares estimators the sum of the squares $SS = \sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2$ is minimised and the error terms are assumed to be normal. It is a simple calculation to show that the least squares estimators are also maximum likelihood estimators. Outliers will have a pronounced effect on the sum of the squares because of the square function. If the error terms are known not to be normal or an abnormal number of outliers are observed then the estimates can be improved either by transforming the data, using maximum likelihood estimates or using robust estimators.

Consider the case where a linear model has been fit to data and the error when plotted against the original x values appear as in *Exhibit 19.16* below.

Exhibit 19.16

Error Terms

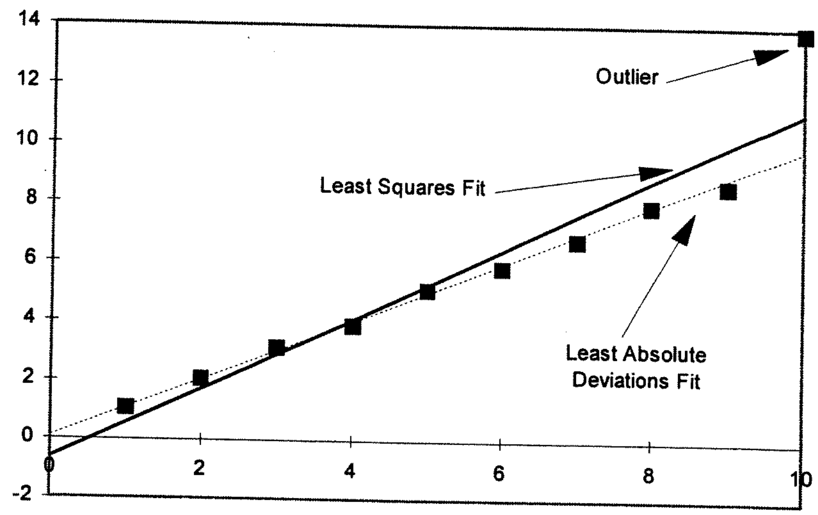


The increasing error magnitude as shown in *Exhibit 19.16* above indicates that the linear model is not a good description of the data and maybe a log transform may be appropriate.

If the error terms are known to be of a certain distribution then the ordinary least squares estimators can be replaced by the appropriate maximum likelihood estimators. The likelihood is simply a function of the known density function and the observed data. The estimates may have to be estimated numerically (see section 6.6 for a brief discussion of numerical methods).

The effect of outliers on a least squares fit are pronounced, for example consider the bulk of the data in *Exhibit 19.17* below and the outlier.

Exhibit 19.17
Robust Regression



Some authors advocate the use of least absolute deviations as a robust methodology to fit this kind of data. Here the sum of the absolute deviations

$$SAD = \sum_{i=1}^{n} |y_i - \hat{y}_i|$$

is minimised with respect to the parameters of the model. In this way outliers have considerably less impact on the result. *Exhibit 19.17* illustrates this point.

An alternative sometimes used is weighted least squares. Each point is given a weight w_i and the weighted sum of squares is minimised

$$WSS = \sum_{i=1}^{n} w_i (y_i - \hat{y}_i)^2$$

The determination of the weights is non-unique and may be adaptive. For a more thorough discussion of robust parameter estimation readers are referred to Huber[36] or Launer and Wilkinson.[37]

6.5 Constrained optimisation and the method of Lagrange multipliers

Sections 6.1 to 6.4 above are examples of optimisation. In section 5.1 the concept of the "sum of the squares" was introduced and it was shown how this was minimised using calculus.

36. Op cit n 27.
37. Op cit n 29.

In most practical cases it is desirable to constrain the parameters or a linear function $f(\cdot)$ of the parameters. Constraints can be either *equality constraints* such as $f(\mathbf{X}) = c$ or *inequality constraints* such as $f(\mathbf{X}) \le c$. Such a constraint in the simple linear regression example given above would be to constrain the β parameter to be greater than or equal to zero.

For equality constraints the method of *Lagrange multipliers* is useful. If the original problem involves minimising the *objective function* $h(\mathbf{X})$, the constrained problem can be solved by defining the *Lagrangian* as

$$L = h(\mathbf{X}) - \lambda(f(\mathbf{X}) - c)$$

and solving the equations

$$\frac{\partial L}{\partial \mathbf{X}} = 0 \text{ and } \frac{\partial L}{\partial \lambda} = 0$$

The new parameter λ is called a *Lagrange multiplier*. If more than one equality constraint is required, these equations may be extended in the logical manner.

Consider the example where we wish to replicate an equity index such as the Australian all ordinaries index with two stocks from the index such as BHP and CRA. Letting Y be the index return, x_1 be the BHP return and x_2 be the CRA return. The model will be of the form

$$Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$$

The parameters β_1 and β_2 can be estimated using the method of least squares and some data. Using the methodology discussed in sections 6.1 and 6.2 above, these estimators will satisfy the equations

$$\begin{pmatrix} \sum_{i=1}^{n} x_{1i}^2 & \sum_{i=1}^{n} x_{1i}x_{2i} \\ \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{2i}^2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_i x_{1i} \\ \sum_{i=1}^{n} y_i x_{2i} \end{pmatrix}$$

These equations may easily be solved for the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$.

If it is desired to constrain the solution so that the portfolio is fully invested in the market, that is $\beta_1 + \beta_2 = 1$. The Lagrangian is now

$$L = \sum_{i=1}^{n} \left(y_i - \left(\beta_1 x_{1i} + \beta_2 x_{2i}\right)\right)^2 - \lambda\left(\beta_1 + \beta_2 - 1\right)$$

The three equations that now must be solved are

$$\frac{\partial L}{\partial \beta_1} = 0 \Rightarrow \hat{\beta}_1 \sum_{i=1}^{n} x_{1i}^2 + \hat{\beta}_2 \sum_{i=1}^{n} x_{1i}x_{2i} - \frac{\lambda}{2} = \sum_{i=1}^{n} y_i x_{1i}$$

$$\frac{\partial L}{\partial \beta_1} = 0 \Rightarrow \hat{\beta}_1 \sum_{i=1}^{n} x_{1i}x_{2i} + \hat{\beta}_2 \sum_{i=1}^{n} x_{2i}^2 - \frac{\lambda}{2} = \sum_{i=1}^{n} y_i x_{2i}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \hat{\beta}_1 + \hat{\beta}_2 = 1$$

Notice that the last equation guarantees the constraint will be satisfied by the solution. In matrix notation these equations are

$$\begin{pmatrix} \sum_{i=1}^{n} x_{1i}^{2} & \sum_{i=1}^{n} x_{1i}x_{2i} & -\frac{1}{2} \\ \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{2i}^{2} & -\frac{1}{2} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_i x_{1i} \\ \sum_{i=1}^{n} y_i x_{2i} \\ 1 \end{pmatrix}$$

which may be easily solved using matrix inversion.

For inequality constraints, the method of Lagrange multipliers may be extended with an iterative scheme to alternatively satisfy the constraints.[38]

6.6 Linear, quadratic and integer programming

Consider the more general problem

minimise $h(\mathbf{X})$

subject to $f_i(\mathbf{X}) \le c_i$ for $i = 1 \ldots p$

$f_i(\mathbf{X}) = c_i$ for $i = p + 1 \ldots p + q$

If the *objective function* $h(\mathbf{X})$ and the *constraint functions* $f_i(\mathbf{X})$ are linear the problem is called a *linear programming* problem. The *simplex method* is used to solve such problems. Such problems are often a good approximation to a more complex problem.

If the objective function $h(\mathbf{X})$ is quadratic in $\mathbf{X}$ (that is, it can be expressed as $h(\mathbf{X}) = \sum_{i,j} a_{ij}x_i x_j + \sum_i b_i x_i$, the problem is one of *quadratic programming*. The example in section 6.5 above was an example of this type.

If a linear problem is further constrained so that the parameters X are to be integers, the problem is one *integer programming*.

These sorts of problems are part of the field of *operations research*. The solutions to these types of problems are reasonably involved and beyond the scope of this discussion, the interested reader is referred to Taha[39] or similar treatments in this field.

More often than not, a numerical approach must be used to solve any realistic problem. Optimisation theory is an active area of research in numerical mathematics, generating new more efficient methods. For a rudimentary and somewhat dated approach, the reader is referred to Press, Flannery, Teukolsky and Vetterling.[40] For an up-to-date but advanced discussion of the implementation of one efficient algorithm see Lawrence, Zhou and Tits.[41]

38. See H Taha, *Operations Research: An Introduction* (5th ed, Macmillan Publishing, 1992), section 18.2.2.
39. Ibid.
40. Op cit n 5.
41. C Lawrence, J Zhou and A Tits, *Users Guide for CFSQP Version 2.4: AC Code for Solving (Large Scale) Constrained Nonlinear (Minimax) Optimisation Problems, Generating Iterates Satisfying All Inequality Constraints* (Electrical Engineering Dept and Institute for Systems Research, University of Maryland, College Park, MD 20742 USA, 1996).

Microsoft Excel incorporates an optimiser, called Solver, as an add in. Solver is easy to use and can handle quite general and complex problems. Solver can analyse linear, nonlinear and integer problems. Solver is more than adequate to prototype most problems but slow to tackle any problem with a large number of dimensions or complex constraints. Refer to the online documentation or the Microsoft Excel User's Guide.

7. RANDOM PROCESSES AND STOCHASTIC CALCULUS

Up until now the discussion of asset price probability distributions has been concerned with one point in the future, typically the expiry date of the option as in section 4.13. More often than not mathematical finance is concerned with how the stock price moves in the intervening time or the path it takes over the time interval. The theory of random processes provides a convenient framework to study these paths

With a European option the value at expiry depends only on the value of the asset at expiry. With a path dependent option, the value depends upon the actual path the asset price takes from its initiation to expiry. The class of path-dependent options includes American, average or Asian, lookback and barrier options. The results of delta hedging an option will also depend upon the path the asset price takes. Section 7.7 covers the classical derivation of the Black-Scholes first presented in Black and Scholes.[42]

More than any other, this section is a summary of the most important points and results from a difficult area of mathematics. Interested readers should consult Ross[43] for an introduction to this topic. For a more thorough and complete coverage of this material the reader is referred to an excellent and topical coverage in Malliaris and Brock.[44] Also of use could be Schuss[45] or Bhattacharya and Waymire.[46] Many other books exist on this subject, but most are of an extremely technical nature.

7.1 Definitions

A *random process* or *stochastic process* is a collection of random variables $\{X(t), t \in T\}$. The index set T may be either discrete or continuous and the random variables $X(t)$ may be either discrete or continuous. If the index set T is discrete then the process is said to be a *discrete time* process and if the index set is continuous then the process is said to be *continuous time* process.

42. Op cit n 11.
43. S Ross, *An Introduction to Probability Models* (4th ed, Academic Press, 1989).
44. A Malliaris and W Brock, *Stochastic Methods in Economics and Finance* (North-Holland, 1982).
45. Z Schuss, *Theory and Application of Stochastic Differential Equations* (Wiley, 1980).
46. R Bhattacharya and E Waymire, *Stochastic Processes with Applications* (Wiley, 1990).

Consider the number of trades in a day for a particular stock, this can be represented as the process $\{N_1,N_2...\}$ where N_1 is a discrete random variable representing the number of trades on the first day, N_2 represents the number of trades on the second day and so forth. This is an example of a discrete time random process of discrete random variables.

Consider the price of a particular stock over time, this can be represented as the random process $\{S(t),t \geq 0\}$ where $S(t)$ represents the stock price distribution a time t in the future. This is an example of a continuous time random process of continuous random variables. Most of the discussion will focus on these types of random processes as they are a good framework in which to model asset price processes.

One sample $\{x(t),0 \leq t \leq t_{max}\}$ of the random process $\{X(t),t \geq 0\}$ is called a *path* or a *realisation* of the random process.

7.2 Markov processes

A *Markov process* is one where only the present state of the process is important and the actual path taken to reach the present state is unimportant. More formally for $0 \leq s \leq t$

$$\Pr\{X(t) \leq y|X(u) = x(u),0 \leq u \leq s\} = \Pr\{X(t) \leq y|X(s) = x(s)\}$$

That is, a Markov process is one that depends upon the present only, all of the information contained in the past is also contained in the present state of the process.

It is usually assumed that asset prices follow a Markov process. Hull[47] argues strongly that the Markov property of observed processes is a result of market efficiency. Although impossible to empirically prove, the Markov property has been shown to be at least a good first approximation.

7.3 Wiener process

The standard *Wiener process* or *Brownian motion* $\{X(t),t \geq 0\}$ is defined by the properties

1) $X(0) = 0$.
2) $\{X(t),t \geq 0\}$ has stationary and independent increments, that is $X(t + y) - X(t)$ and $X(t + 2y) - X(t + y)$ are independent and have the same distribution.
3) $X(t) \sim N(0,t)$ for all $t > 0$.

Property 2) can be used to show that the Wiener process has the Markov property.

These properties can be used to show that

$$X(t) - X(s) \sim N(0,t - s) \text{ for } 0 \leq s < t$$

The Wiener process was used by Einstein to describe the motions of particles in liquids that are due to a large number of small movements. Mathematical finance also uses the Wiener process to describe the movement of stock prices over time. The statement in section 4.6 claiming that the stock

47. Op cit n 10.

price distribution is normally distributed is equivalent to claiming the stock price process is a Wiener process.

7.4 First passage times and maximum variables

Some useful characteristics of any Wiener process are the time it takes for the process to first reach a level and the maximum of the process over a finite period. Let T_a denote the first time the Wiener process $\{X(t), t \geq 0\}$ exceeds the level $a \geq 0$. Using conditional probability it is easy to show that

$$\Pr\{X(t) \geq a\} = \Pr\{X(t) \geq a | T_a > t\} \Pr\{T_a > t\} + \Pr\{X(t) \geq a | T_a \leq t\} \Pr\{t_a \leq t\}$$

If $T_a \leq t$, then as the process is symmetric, there is just as much chance the process is above a at time t as below a, hence

$$\Pr\{X(t) \geq a | T_a \leq t\} = 1/2$$

Also if $T_a > t$, then it is impossible for $X(t) > a$ and so

$$\Pr\{X(t) \geq a\} = 0.\Pr\{T_a > t\} + 1/2 \Pr\{T_a \leq t\}$$

or

$$\Pr\{T_a \leq t\} = 2\Pr\{X(t) \geq a\}$$
$$= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$$

This distribution is also known as the hitting time distribution. One interesting feature is that $E[T_a] = \infty$, that is, although the process will reach a in finite time, the expected value is infinite.

Let $M(t) = \max_{0 \leq s \leq t} X(s)$ be the maximum level the process reaches over a finite time t, it can easily be shown that

$$\Pr\{M(t) > a\} = \Pr\{T_a \leq t\}$$
$$= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$$

and the expected value is

$$E[M(t)] = \sqrt{\frac{2t}{\pi}}$$

These equations are the foundations for the valuation of lookback[48] and barrier[49] options.

It can also be shown that given $A>0$ and $B>0$ that

$$\Pr\{T_A > T_{-B}\} = \frac{B}{A+B}$$

This is just the probability that the random process will hit A before it passes through $-B$. This equation can also be used to solve the Gambler's

48. An option with a payoff that depends upon the maximum or minimum of the asset price over a fixed period of time.
49. An option that terminates or is initialised if the asset price passes through a certain fixed barrier.

Ruin Problem and evaluating the probability of being "stopped out" of trading positions before a trade is closed out at a profit.

7.5 Stochastic calculus

Consider a Wiener process $\{Z(t), t \geq 0\}$ as introduced by section 6.3. For a small time Δt it can be shown that

$$Z(t + \Delta t) - Z(t) \sim N(0, \Delta t)$$

Alternatively, if $U \sim N(0,1)$ then

$$Z(t + \Delta t) - Z(t) = \sqrt{\Delta t}U$$

Taking the limit $\Delta t \to 0$ this equation may be written

$$dZ(t) = \sqrt{\Delta t}U$$

Notice that the usual expression $\frac{dZ(t)}{dt}$ does not exist, a Wiener process is nowhere differentiable. A more general stochastic process, also known as an *Itô process* may be written as

$$dX(t) = a(t,X)dt + b(t,X)dZ$$

which represents a Wiener process with drift $a(t,X)$ and intensity $b(t,X)$. The assumption of the stock price distribution introduced in section 4.6 as

$$\log\left(S(t)/S(0)\right) \sim N\left(\left(r - \tfrac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

can be rewritten as a stochastic differential equation as

$$\frac{dS}{S} = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dZ$$

This differential equation may be integrated to yield

$$\int_{u=0}^{t} \frac{dS}{S} = \int_{u=0}^{t} \left(r - \tfrac{1}{2}\sigma^2\right)du + \int_{u=0}^{t} \sigma dZ$$

The first integral may be formally integrated to yield

$$\int_{u=0}^{t} \frac{dS}{S} = \log(S(t)) - \log(S(0)) = \log\left(S(t)/S(0)\right)$$

The second integral may also be formally integrated to give

$$\int_{u=0}^{t} \left(r - \tfrac{1}{2}\sigma^2\right)du = \left(r - \tfrac{1}{2}\sigma^2\right)t$$

While the last integral is not of the usual calculus type, but observing that σ is a constant it may be simplified to

$$\int_{u=0}^{t} \sigma dZ = \sigma \int_{u=0}^{t} dZ = \sigma\left(Z(t) - Z(0)\right) = \sigma Z(t)$$

where $\{Z(t), t > 0\}$ is the usual Wiener process. Hence, the stochastic differential equation $\frac{dS}{S} = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dZ$ is identical to the statement about the probability distribution of $S(t)$.

7.6 Itô's lemma

One of the most important result of stochastic calculus is Itô's Lemma. In section 7.7 it will be shown how this lemma is applied to deriving the Black-Scholes equation.

Consider a function $f = f(X(t),t)$ of an Itô process $dX(t) = a(t,X)dt + b(t,X)dZ$ and time. Then Itô's Lemma states that

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X} a(t,X) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} b(t,X)^2\right)dt + \frac{\partial f}{\partial X} b(t,X)dZ$$

The interesting thing about this equation is the term $\frac{1}{2}\frac{\partial^2 f}{\partial X^2} b(t,X)^2 dt$. Consider a two dimensional Taylor expansion of $f(X(t,t)$

$$f(X+\Delta X, t+\Delta t) - f(X,t) = \frac{\partial f}{\partial X}\Delta X + \frac{\partial f}{\partial t}\Delta t$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^2}\Delta X^2 + \frac{\partial^2 f}{\partial X \partial t}\Delta X \Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2$$
$$+ \text{ higher order terms}$$

By replacing Δt with dt and ΔX with dX this equation simplifies to

$$df = \frac{\partial f}{\partial X}(adt + bdZ) + \frac{\partial f}{\partial t}dt$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^2}(a^2dt^2 + 2abdtdZ + b^2dZ^2) + \frac{\partial^2 f}{\partial X \partial t}(adt^2 + bdtdZ) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2$$
$$+ \text{ higher order terms}$$

Notice the term in dZ^2, we know from the properties of the standard Wiener process and the normal distribution that $E[dZ^2] = dt$ and $Var[dZ^2] = 2dt^2$, hence in the limit dZ^2 becomes a constant value of dt. Taking the limit $dt \to 0$, only terms in dt and dZ are significant. This yields the lemma.

7.7 Black-Scholes equation revisited

Using Itô's Lemma it is a simple matter to derive the Black-Scholes equation. Using the assumption above that the instantaneous change in value of the stock S over time t is governed by Itô process

$$\frac{dS}{S} = \left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dZ$$

Consider the value of a European call option[50] $C_{K,t}(S,t)$ with strike K and time to expiry t, as only a function of the stock price S and time. Itô's lemma shows that $C_{K,t}(S,t)$ is governed by the stochastic differential equation

$$dC = \left(\frac{\partial C}{\partial S}\left(r - \tfrac{1}{2}\sigma^2\right)S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial C}{\partial S}\sigma dW$$

50. A European call option is the right but not the obligation to purchase the underlying asset at a fixed price (strike price), on a certain date (expiry date).

We construct a portfolio P, of long (purchase) of 1 call option and short (sale) of δ shares

$$P = C - \delta S$$

Then the instantaneous change in value of the portfolio is given by

$$\begin{aligned} dP &= dC - \delta dS \\ &= \left(\frac{\partial C}{\partial S}\left(r - \tfrac{1}{2}\sigma^2\right)S + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 - \delta S\left(r - \tfrac{1}{2}\sigma^2\right)\right)dt \\ &+ \left(\frac{\partial C}{\partial S} - \delta\right)S\sigma dW \end{aligned}$$

If δ is chosen such that

$$\delta = \frac{\partial C}{\partial S}$$

this choice implies

$$P = C - \frac{\partial C}{\partial S}S \qquad and$$

$$dP = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt$$

Then the portfolio is riskless and so its return must be the risk free interest rate r and so

$$dP = rPdt$$

Thus we have derived the celebrated *Black-Scholes differential equation* for the value of a call option

$$\frac{\partial C}{\partial t} + r\frac{\partial C}{\partial S}S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 = rC$$

This second order linear parabolic partial differential equation must be solved with regard to the boundary condition

$$C_{K,t} = \max(S - K, 0) \text{ at } t = T$$

It can be shown that solution

$$\begin{aligned} C_{K,t} &= S_0\Phi\left(\frac{\log\left(S_0/K\right) + \left(r + \sigma^2/2\right)t}{\sigma\sqrt{t}}\right) \\ &- e^{-rt}K\Phi\left(\frac{\log\left(S_0/K\right) + \left(r - \sigma^2/2\right)t}{\sigma\sqrt{t}}\right) \end{aligned}$$

satisfies the boundary condition and the Black-Scholes partial differential equation. Note that $\Phi()$ is the standard cumulative normal distribution.

Note that the Black-Scholes partial differential equation holds for all types of securities; the only difference is the boundary conditions change. For example, exactly the same partial differential equation holds for European put option,[51]

$$\frac{\partial P}{\partial t}+r\frac{\partial P}{\partial S}S+\frac{1}{2}\frac{\partial^2 P}{\partial S^2}\sigma^2 S^2 = rP$$

but the boundary condition has changed to

$$P_{K,t} = \max(K - S,0) \text{ at } t = T$$

This solution is important because it shows how a riskless portfolio of options and underlying asset may be constructed. For more advanced options it is not always so easy to solve the partial differential equation; numerical methods are often resorted to.[52] Note also that this method does not rely on the drift of the stochastic process being $r - {}^1/_2\sigma^2$ as the risk neutral derivation of section 4.13 did.

8. SIMULATIONS

Simulation, or Monte Carlo, techniques are an important tool in the valuation of financial securities. For many types of exotic derivatives, no closed form solution for the value exist. In these cases either the partial differential equation needs to be solved or a Monte Carlo technique may be adopted. Simulation can also be used to measure the effect of changing the underlying stochastic process.[53]

One of the Black-Scholes[54] assumptions is that the option may be continuously hedged. This is plainly unrealistic as markets tend to move very quickly (gap) and the transaction costs associated with this scheme would be prohibitive. Monte Carlo methods are useful to estimate the effect of different hedge methodologies.

The term "Monte Carlo" refers to a broad class of techniques that use random numbers to simulate the possible outcomes of a system. In mathematical finance, typically the random stock or asset price is assumed to be normally distributed and so normal random numbers are used to simulate the possible asset price at some point in time. From these asset prices the value of the derivative security may be easily evaluated and so the long run average may be established if enough sampling is performed. Mathematically, the basic principal of Monte Carlo simulation is to approximate the expected value E[$f(X)$] with $\frac{1}{n}\sum_{i=1}^{n} f(x_i)$ where x_i are samples from the same distribution as X. Moreover, the Central Limit Theorem tells us, under a few regulatory conditions, that

51. A European put option is the right but not the obligation to sell the underlying asset at a fixed price (strike price), on a certain date (expiry date).
52. See P Wilmott, J Dewynne and S Howison, *Option Pricing: Mathematical Models and Computation* (Oxford Financial Press, 1994).
53. For example see J Hull and A White, "The Pricing of Options on Assets with Stochastic Volatilities" (1987) 42 (June) *Journal of Finance* 281.
54. Black and Scholes, op cit n 11.

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \xrightarrow[n\to\infty]{d} N\left(\mathrm{E}[f(X)], \frac{\mathrm{Var}[f(X)]}{n}\right).$$

Note that the standard deviation of the estimate is of order $\frac{1}{\sqrt{n}}$. This criteria will determine how many samples is enough to estimate the average within a specified tolerance.

8.1 Generation of uniform deviates

Generation of random deviates from the uniform distribution (section 4.5 above) is the starting point for generating other types of deviates. The reason is simple to justify, because if X is a random variable with cumulative distribution function $F(\cdot)$, then $F(X)$ is distributed as a uniform random variable. Hence if U is a uniform random variable then $F^{-1}(U)$ is a random variable with cumulative distribution function $F(\cdot)$. This is called the *transform method* of generating random deviates from a known distribution.

Most programming languages have routines built in to generate uniform deviates but care must be taken with these routines. Most built in generators use the linear congruential method calculated by the integer recurrence formula

$$I_{j+1} = (aI_j + b) \bmod m$$

where a, b and m are pre-set integers. This calculation will be extremely fast but only generate m distinct values. This method can be quite unacceptable if the implementation uses small integers or the three constants a, b and m are poorly chosen.[55] In short, do not use these routines for important calculations.

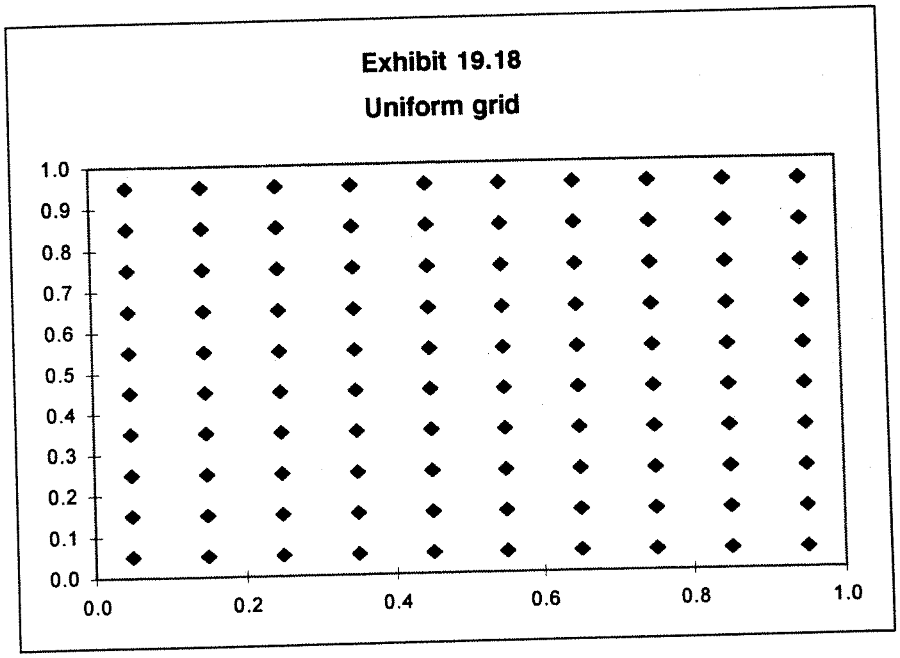
Many portable random number generators that produce high quality and fast uniform deviates exist, see Press, Flannery, Teukolsky and Vetterling,[56] or for a more up to date discussion see Tezuka.[57]

Microsoft Excel provides a built in uniform random number generator RAND. Exercise care.

55. For a discussion of these problems see D Knuth, *The Art of Computer Propgramming: Seminumerical Algorithms* (Vol II, 2nd ed, Addison-Wesley, 1981) or Press, Flannery, Teukolsky and Vetterling, op cit n 5.
56. Op cit n 5.
57. S Tezuka, *Uniform Random Numbers: Theory and Practice* (Kluner Academic Publishers, 1995).

8.2 Sobol quasi random numbers

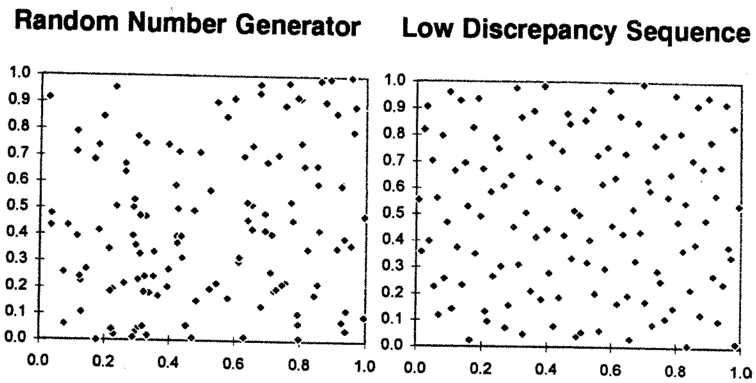
As discussed above, the convergence of Monte Carlo techniques is of order $\frac{1}{\sqrt{n}}$. As an alternative consider using n equally spaced points on a grid. Such an example would be to choose 100 points on the two dimensional plane as in *Exhibit 19.18* below.



Using this scheme, the sum $\frac{1}{n}\sum_{i=1}^{n} f(x_i)$ will approximate the integral E[$f(X)$] with an error of $\frac{1}{n}$, a worthwhile improvement for large n. The problem with this type of scheme is that, once started, the whole scheme must be completed, otherwise a biased sample will be taken.

The aim of quasi random or low discrepancy sequences is to choose points in the p dimensional space as uniformly as possible. With these methods, each successive point is chosen to be "as far away" from the previous points as possible. In this way the clustering of points observed with random number generators is avoided. *Exhibit 19.19* shows 127 points generated with a uniform random number generator and also with a low discrepancy sequence.

Exhibit 19.19



There is more than one way of generating these low discrepancy sequences. Common types are Sobol, Halton and Faure sequences. Each differ markedly in their operation and their asymptotic properties. For example, the Sobol sequence used to generate *Exhibit 19.19* above has a convergence rate of $\frac{(\log n)^p}{n} \underset{n\to\infty}{\equiv} \frac{1}{n}$.

The derivation of these sequences is based on advanced number theory and is beyond the scope of this chapter. For a detailed discussion and an example of an implementation of a Sobol sequence see Press, Flannery, Teukolsky and Vetterling,[58] or for a more up-to-date discussion see Tezuka.[59] For an example of how these sequences have been applied to mathematical finance see Papageorgiou and Traub.[60]

8.3 Generation of standard normal deviates

For financial applications, the generation of normally distributed random variables is important. Once standard normal deviates are generated and rescaled, stock prices may be simulated using the usual normal assumption (see section 4.6).

Some authors[61] advocate the use of the formula

$$X = \sum_{i=1}^{12} U_i - 6, \quad U_i \sim Uniform$$

to generate normally distributed random deviates. It does not take long to see that this method is fairly crude and slow. In addition, this type of estimate would be incompatible with Sobol or quasi random numbers, using the routine 12 times to generate one normal deviate would destroy the low discrepancy behaviour of the routine.

58. Op cit n 5.
59. Op cit n 56.
60. A Papageorgiou and J Traub, "Beating Monte Carlo" (1996) 9(6) *Risk Magazine*.
61. See Hull, op cit n 10, for example.

A common alternative is to use the Box-Muller method of generating standard normal random numbers. If $U_1$and U_2 are uniform deviates then

$$Z_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$
$$Z_1 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$$

are normally distributed. There is an elegant explanation of these transformations. Consider two independent standard normal variables specifying the horizontal and vertical distance from the origin of a point on the two-dimensional plane. It is easy to see that the angle the line joining the point with the origin makes with the horizontal axis will be distributed Uniform[0,2π]. Furthermore, the distance squared of the point from the origin will have cumulative distribution function $F(x) = 1 - {}^1\!/_2 e^{-x/2}$. This last function may be easily inverted using the transform method discussed above. While elegant in derivation, this method is reasonably slow[62] to calculate and cannot be used with Sobol sequences.

A further alternative is to use the transform method by inverting the normal cumulative distribution function directly. Just as there is no closed for solution for the normal cumulative distribution function there is no closed form solution for its inverse. Fortunately, this function can be approximated by a function of two polynomials. See Abramowitz and Stegun,[63] or Moro[64] for a particularly good example. This type of approach can be used with Sobol and low discrepancy sequences.

62. A slight modification exists to avoid the computation of the two trigonometric functions, see Press, Flannery, Teukolsky and Vetterling, op cit n 5, or Ross, op cit n 43.
63. Op cit n 9.
64. B Moro, "The Full Monte" (1995) 8(2) *Risk Magazine*.

Appendix I

Glossary

Bivariate distribution

A probability distribution that describes the distribution of two, not necessarily independent, random variables.

Brownian motion

See **Wiener process**.

Consistent estimator

An **estimator** whose variance tends to zero if enough samples are taken.

Estimator

A function or algorithm based on an observed sample that gives a numerical value to a parameter of the underlying system.

Historical volatility

An estimate of the annualised standard deviation of asset returns based upon an observes time series of asset prices.

Lagrange multiplier

A dummy variable introduced in constrained optimisation to reflect an equality constraint.

Linear programming

A constrained optimisation problem where the objective function is linear in the unknown parameters.

Markov process

A **stochastic process** where the increments are independent.

Matrix

Rectangular array of numbers.

Monte Carlo

A method where the random variable is numerically simulated and so the long run average of a function of the random variable may be evaluated.

Normal distribution

The classical "bell shaped" distribution that underlies most of statistics because of the central limit theorem.

Outlier

A lone observation that deviates markedly form the behaviour of the rest of the data.

Quadratic programming

A constrained optimisation problem where the objective function is quadratic in the unknown parameters.

Random process

See **Stochastic process.**

Random variable

A mathematical concept for a variable that has no fixed value, but can be described by a probability distribution. Random variables may take on either discrete values or continuous values.

Robust estimator

An **estimator** that is not overly influenced by outliers.

Simplex method

A procedure used to solve a **linear programming** problem.

Stochastic process

A set of random variables, indexed by time, that describes the evolution of a probabilistic system. Also known as a random process.

Sobol pseudo random sequence

A sequence of numbers chosen to be as far away form the previous numbers in the sequence. Used in **Monte Carlo** techniques to speed convergence. Also known as *low discrepancy sequences*.

Unbiased estimator

An **estimator** whose mean is the parameter to be estimated.

Vector

A **matrix** with either one row or one column.

Wiener process

A **stochastic process** whose increments are independent and normally distributed. Also known as *Brownian motion*.

Appendix II

Summary of Notation and Symbols

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$	The 2 × 2 matrix with entries $a_{11}, a_{12}, \ldots$
$[\mathbf{A}]_{ij}$	The element of the matrix **A** in the ith row and jth column.
$\mathbf{A}^{-1}$	Inverse of the matrix **A**.
$\mathbf{A}^T$ or $\mathbf{A}'$	The transpose of the matrix **A**.
${}^nC_r$ or $\binom{n}{r}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$
Corr[X,Y]	The correlation of two random variables X and Y.
Cov[X,Y]	The covariance of two random variables X and Y.
$\frac{\partial f(x,y)}{\partial x}$	The partial derivative of $f(x,y)$ with respect to x.
$\frac{df(x,y)}{dx}$	The total derivative of $f(x,y)$ with respect to x.
exp(x) or e^x	The exponential function where e = 2.71828...
E[$g(X)$]	Expected value of a function $g(X)$ of a random variable X.
!	Factorial that is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. Note $o! = 1$.
$\Gamma(x)$	The gamma function, that is $\Gamma(x) = \int_{u=0}^{\infty} u^{x-1} e^{-u} du$.
I	The identity matrix.
$\int f(x) dx$	The indefinite integral of $f(x)$ with respect to x.
$\int_{x=y}^{z} f(x) dx$	The definite integral of $f(x)$ with respect to x from $x = y$ to $x = z$.
$\lim_{x \to 0} f(x)$	The limit of $f(x)$ as x tends to zero.
log(x)	The natural logarithm function, the inverse of e^x, that is $\log(e^x) = e^{\log(x)} = x$.
$N(\mu,\sigma^2)$	A normal distribution with mean μ and variance σ^2.
$\Phi(x)$ or $N(x)$	The cumulative normal distribution $\Phi(x) = N(x) = \int_{z=-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$.
Pr{A}	Probability of an event A.
Pr{B|A}	Conditional probability of event B given A.

$\prod_{i=1}^{n} a_i$	The product $a_1 \cdot a_2 \cdot \ldots \cdot a_n$.
$\sum_{i=1}^{n} a_i$	The summation $a_1 + a_2 + \ldots + a_n$.
Var[X]	The variance of a random variable X.
$\sim$	Distributed as.

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Entries in italics refer to illustrations